Prime Obsession Notes

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Abstract

Despite mathematicians best efforts, a proof for Riemann's famous 1859 hypothesis has proved elusive for over a century and a half. John Derbyshire published his *Prime Obsession* [1] as a way to demystify the basics of Riemann's conjecture so that amateurs and enthusiasts can begin to grasp the work that has been undertaken to solve this fascinating problem. The basic statement of the problem is as follows: all non-trivial zeros of the zeta function have real part one-half. The Zeta Function is denoted $\zeta(s)$ and has complex arguments s mapping to $\displaystyle\sum_n \frac{1}{n^s}$.

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1 Properties of Complex Numbers

1.1 Complex Plane and Function Maps

We call $z \in \mathbb{C}$ a complex number if it can be written in the form $z = a+bi$ where a and b are real numbers and $i^2 = -1$. The real part of z, denoted $Re(z)$, is entirely determined by a while the imaginary part of z, denoted $Im(z)$, is entirely determined by b. To plot z on the complex plane, we let $Re(z)$ run horizontally and $Im(z)$ run vertically with i as the unit scaling factor.

It then necessitates four dimensions to visualize a function $f : \mathbb{C} \to \mathbb{C}$; two for the argument plane and two for the value plane. This type of depiction is unnatural. For example, in four dimensional space, two flat planes can intersect at just a single point (just like two non-parallel lines need not intersect in three-space, unimaginable in two-space). Rigorous definitions of space and dimension are provided in previous papers dealing with linear algebra [2] [3].

Instead, we elect to show separate argument and value planes. To do this, we will mark where interesting values (namely zero's or axis points of a function) are mapped to (in the case of the value plane), or where these interesting values come from (in the case of the argument plane).

For example, in Figure 1.1 the green lines represent points that are mapped to either the real or complex axis and red dots represent the zero's of the Zeta Function. Notice that all the zero's of the function seem to be either even negative real numbers (which are called trivial zeros) or else have real part $\frac{1}{2}$ (the non-trivial zeros). The Riemann Hypothesis is that every non-trivial zero lies on this "critical line".

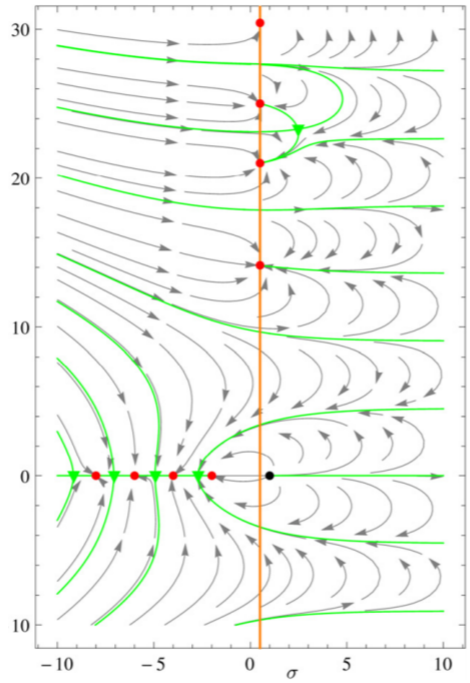

Figure 1.1: Argument Plane of the Zeta Function [5]

Meanwhile in Figure 1.2, we can see where these points on the critical line are mapped to.

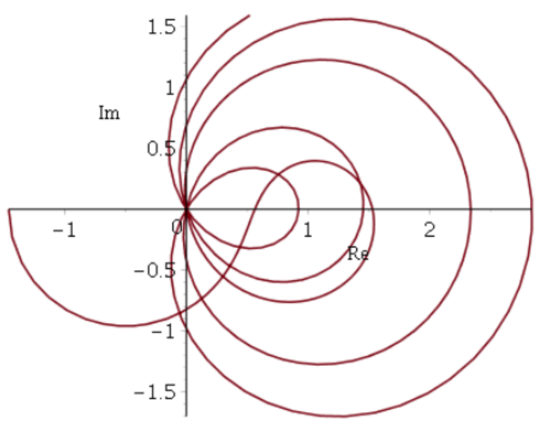

Figure 1.2: Value Plane of the Zeta Function [6]

1.2 Arithmetic and Geometry of Complex Numbers

Definition 1.1. The magnitude of a complex number $z = a + bi$, denoted $|z|$, is the Euclidean Distance from z to the origin of the complex plane. It is calculated as $|z| = \sqrt{a^2 + b^2}$.

Definition 1.2. The argument of a complex number $z = a + bi$, denoted $Arg(z)$ with $-\pi < Arg(z) \leq \pi$, is the angle (in radians) a point z on the complex plane forms with the real line.

Since the "opposite" (to use common trigonometry terms) is determined by b and the "adjacent" is a, $Arg(z)$ can be calculated as $Arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$. Together with the magnitude, the argument is essentially the polar coordinates of the Cartesian Point (a, b).

Definition 1.3. The complex conjugate of a complex number $z = a + bi$, denoted $\bar{z}$, is its reflection across the real axis: $\bar{z} = a - bi$.

The arithmetic of complex numbers is largely as expected. Let $z = a + bi$ and $y = c + di$. Then we see the following: $z + y = (a + c) + (b + d)i$, $z - y = (a - c) + (b - d)i$, and $zy = (ac - bd) + (bc + ad)i$.

To divide, one must first convert the dividend to a real number using the complex conjugate: $\frac{z}{y} = \frac{z\bar{y}}{y\bar{y}} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right) i$.

2 Prime Number Theorem

Let $\pi(n)$ denote the number of prime numbers up to the natural number n. It is not too difficult a task to write a functional script for the first billion or so numbers in the domain. In multiples of 10, one can see the values of the prime counting function below.

```
> factors=function (x) {
+   a=1:(ceiling(x/2))
+   b=a[x%%a==0]
+   return(append(b,x))
+ }
>
> primes=function (x) {
+   i=1; a=vector()
+   for (j in 1:x) {
+     if (length(factors(j))==2) {
+       a[i]=j
+       i=i+1
+     }
+   }
+   return(a)
+ }
>
> x=100; px=vector()
> for (i in 1:(x/10)){
+   px[i]=length(primes(i*10))-1
+ }
> PCT=data.frame(seq(from=10,to=100,by=10), px)
> names(PCT)=c("Input","Output")
> PCT
   Input Output
1     10      4
2     20      8
3     30     10
4     40     12
5     50     15
6     60     17
7     70     19
8     80     22
9     90     24
10   100     25
```

Figure 2.1: A Crude R Script For The Prime Counting Function

An alternative is to use an online source like dcode[4]. To better grasp the distribution, see what happens when we increase the magnitude of the argument and then adjust its scale.

n	$\pi(n)$	$n/\pi(n)$
1,000	168	5.95
1,000,000	78,498	12.74
1,000,000,000	50,847,534	19.67
1,000,000,000,000	37,607,912,018	26.59
1,000,000,000,000,000	28,844,570,422,669	33.51

Figure 2.2: Prime Counting Function At Multiples of 1000

This resembles a logarithmic function– as the inputs increase multiplicatively, the outputs increase additively. It is natural to map $\log_e n$ along with the prior observations, which is done in Table 1.1 below. As is standard, further references to the natural logarithm will be abbreviated log.

n	$\pi(n)$	$n/\pi(n)$	$\ln(n)$	% error
1,000	168	5.95	6.91	16.05
1,000,000	78,498	12.74	13.82	8.45
1,000,000,000	50,847,534	19.67	20.72	5.37
1,000,000,000,000	37,607,912,018	26.59	27.63	3.91
1,000,000,000,000,000	28,844,570,422,669	33.51	34.54	3.08

Figure 2.3: Logarithmic Distribution of Prime Numbers

It seems logical to conclude that there is some relation between $\frac{n}{\pi(n)}$ and $log(n)$. Indeed, it is presented without proof that as n tends towards infinity, the error terms between the two functions tends towards zero.

Theorem 2.1. The Prime Number Theorem (PNT): $\pi(n) \sim \frac{n}{\log(n)}$

There are two immediate corollaries to the Prime Number Theorem:

Corollary 2.1.1. The n^{th} prime is approximately $n \log(n)$.

Pf: Let $\{1, k\}$ be a sequence of natural numbers with c primes. Then, on average, the first prime will be $p_1 = \frac{k}{c}$, the second prime will be $p_2 = \frac{2k}{c}$, and the c^{th} prime will be $p_c = \frac{ck}{c} = k$. But assuming the PNT, we know that for large k, there are actually $\frac{k}{log(k)}$, not k primes. So in general the n^{th} prime in the sequence will not be $p_n = \frac{nk}{c}$, but $\frac{nk}{k/\log(k)} = n\log(k) \backsimeq n\log(n)$ for large n. □

Corollary 2.1.2. The probability an integer n is prime is about $\frac{1}{log(n)}$.

Pf: For large N, there are approximately $\frac{n}{\log(n)}$ primes, so the average distribution is about $\frac{n}{\log(n)} \cdot \frac{1}{n} = \frac{1}{\log(n)}$. □

3 Logarithmic Integral Function

The Logarithmic Integral Function is $Li(x) = \displaystyle\int_0^x \frac{1}{\ln t} \mathrm{d}t$, or more precisely $Li(x) = \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\ln t} + \int_{1+\varepsilon}^{x} \frac{\mathrm{d}t}{\ln t} \right)$, accounting for the singularity at $x = 1$ (recall that the logarithm function is undefined at numbers less than or equal to zero, and that a definite integral need not be defined at it's endpoints).

To grasp this function, we first plot the integrand, and observe the vertical asymptote at $t = 1$. Since the log function grows slowly, it makes sense that the inverse log decreases rapidly.

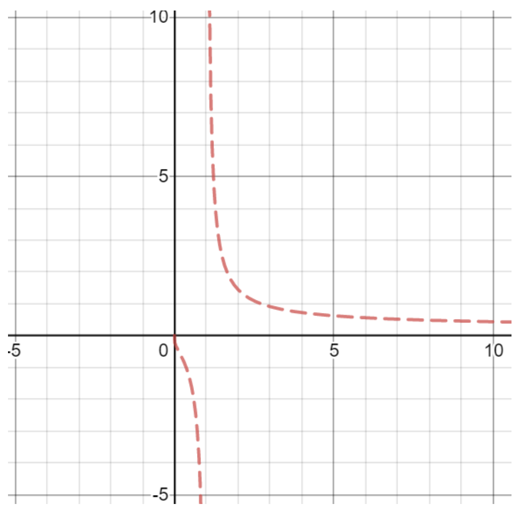

Figure 3.1: Inverse Natural Logarithm [7]

When we integrate the function, values first tend to negative infinity when $x \leq 1$ before switching back positive. This is seen in Figure 3.2 below.

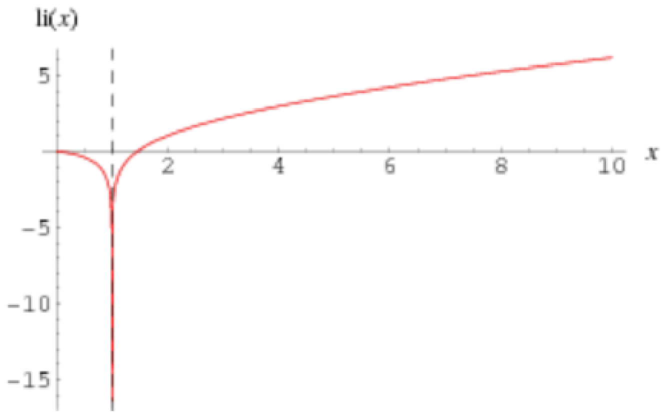

Figure 3.2: Logarithmic Integral [8]

It was Dirichlet who first conjectured that the logarithmic integral provided a better approximation for the prime counting function than the result shown in Section 2.1. Figure 3.3 shows these two approximations.

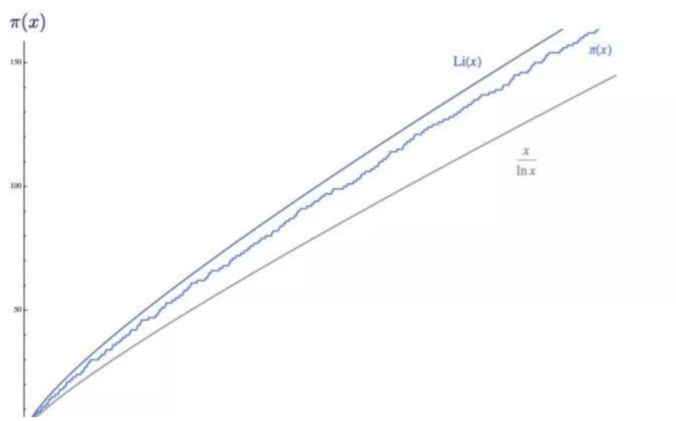

Figure 3.3: Prime Number Theorem [9]

While it appears that $Li(x)$ will consistently overestimate $\pi(x)$ and $\frac{x}{\log x}$ will consistently underestimate it, Littlewood proved in 1914 that $Li(x) - \pi(x)$ alternates between positive and negative values an infinite amount of times. The first instance where $Li(x) \leq \pi(x)$ was originally thought to have an upper bound at Skewes' Number, the massive $e^{e^{e^{e^{7.705}}}}$.

4 Euler's Product Formula

The Zeta Function is:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \dots \tag{4.1}$$

Multiplying by the largest non-unit term $\frac{1}{2^s}$:

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \frac{1}{14^s} + \dots \tag{4.2}$$

Subtracting Equation 4.2 from Equation 4.1:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots \tag{4.3}$$

Multiplying by the largest remaining non-unit term $\frac{1}{3^s}$:

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \frac{1}{39^s} + \dots \tag{4.4}$$

Subtracting Equation 4.4 from 4.3:

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \dots \tag{4.5}$$

Continuing in this fashion, the denominators on the left-hand side will be all of the prime numbers (this is the definition of prime; the subtraction step eliminates any terms with prior factors). Meanwhile, the non-unit terms on the right-side get smaller and smaller (and in fact tend towards zero as the denominators on the left-side tend towards infinity). We are left with:

$$\cdots\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1 \qquad (4.6)$$

Dividing the equation by each term on the left to isolate $\zeta(s)$:

$$\zeta(s)=\left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\left(\frac{1}{1-\frac{1}{7^s}}\right)\left(\frac{1}{1-\frac{1}{11^s}}\right)\cdots \qquad (4.7)$$

Theorem 4.1. Euler's Product Formula: $\zeta(s)=\sum_n \frac{1}{n^s}=\prod_p \left(1-p^{-s}\right)^{-1}$

This is remarkable at first glance. There is an unexpected relationship between the addition of natural numbers and multiplication of prime numbers that involves the Zeta Function and Riemann's famed hypothesis.

5 Möbius Function

5.1 Derivation From Inverse Euler Product

Recall that the product of any m binomials is the sum of all the unique 2^m products of m terms. For instance, $(a+b)(c+d) = (ac)+(ad)+(bc)+(bd)$ and $(a+b)(c+d)(e+f) = (ace)+(acf)+(ade)+(adf)+(bce)+(bcf)+(bde)+(bdf)$. In the second case, the product of three binomials is the sum of eight products of three terms (after selecting the first term in the first binomial, one has a choice of 2 from the next binomial, and then 2 from the last binomial before doing the same process for the second term in the first binomial).

We can say $\dfrac{1}{\zeta(s)} = \prod_p \left(1 - p^{-s}\right)$ from Section 4.1, which is a product of binomials. Using the above logic, we can translate this infinite product into an infinite sum by systematically selecting certain terms from each unique binomial to multiply.

We originally have:

$$\prod_p \left(1 - p^{-s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{11^s}\right)\dots \quad (5.1)$$

We elect for the first term in the sum to be the product of all 1's $(1 \cdot 1 \cdot 1 \cdots)$, the second term in the sum to be the product of all 1's except one term $\left(\frac{-1}{2^s} \cdot 1 \cdot 1 \cdots\right)$, the third term to be the product of all 1's except one different term $\left(1 \cdot \frac{-1}{3^s} \cdot 1 \cdots\right)$, and so on. This is:

$$1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} - \frac{1}{29^s} - \dots \quad (5.2)$$

We then elect for the next terms in the sum to be a product of all 1's expect for two terms. The first term in this infinite sum will be $\left(\frac{-1}{2^s} \cdot \frac{-1}{3^s} \cdot 1 \cdot 1 \cdots\right) = \frac{1}{6^s}$, the next term will be $\left(\frac{-1}{2^s} \cdot 1 \cdot \frac{-1}{5^s} \cdot 1 \cdots\right) = \frac{1}{10^s}$, and so on until we can write $\frac{1}{6^s} + \frac{1}{10^s} + \frac{1}{14^s} + \frac{1}{22^s} + \frac{1}{26^s} + \dots$. After finishing with the $\frac{-1}{2^s}$'s we can move on to the $\frac{-1}{3^s}$'s. We have $\frac{-1}{3^s} \cdot \frac{-1}{5^s} = \frac{1}{15^s}$, $\frac{-1}{3^s} \cdot \frac{-1}{7^s} = \frac{1}{21^s}$, and so on until we can write $\frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{33^s} + \frac{1}{39^s} + \ldots$. This same process will continue for the $\frac{-1}{5^s}$'s, $\frac{-1}{7^s}$'s, and all the other remaining primes. Arranging these infinite sums in decreasing magnitude and combining them with Equation 5.2, we now have:

$$\begin{aligned} &1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} - \ldots \\ &+ \frac{1}{6^s} + \frac{1}{10^s} + \frac{1}{14^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{22^s} + \frac{1}{26^s} + \frac{1}{33^s} + \frac{1}{35^s} + \frac{1}{38^s} + \ldots \end{aligned} \tag{5.3}$$

Continuing in this fashion for all the three-combinations, we first have $\left(\frac{-1}{2^s} \cdot \frac{-1}{3^s} \cdot \frac{-1}{5^s}\right) = \frac{-1}{30^s}$, followed by $\left(\frac{-1}{2^s} \cdot \frac{-1}{3^s} \cdot \frac{-1}{7^s}\right) = \frac{-1}{42^s}$, followed by $\left(\frac{-1}{2^s} \cdot \frac{-1}{3^s} \cdot \frac{-1}{11^s}\right) = \frac{-1}{66^s}$, etc. After completing all the prime multiples of $\left(\frac{-1}{2^s} \cdot \frac{-1}{3^s}\right) = \frac{1}{6^s}$, we can move on to the prime multiples of $\left(\frac{-1}{2^s} \cdot \frac{-1}{5^s}\right) = \frac{1}{10^s}$, or $\left(\frac{-1}{2^s} \cdot \frac{-1}{7^s}\right) = \frac{1}{14^s}$, or $\left(\frac{-1}{3^s} \cdot \frac{-1}{11^s}\right) = \frac{1}{33^s}$, etc. Again arranging this infinite sum of infinite sums in decreasing order by magnitude and combining the results with Equation 5.3, we have:

$$\begin{aligned} &1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \ldots \\ &+ \frac{1}{6^s} + \frac{1}{10^s} + \frac{1}{14^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{22^s} + \frac{1}{26^s} + \frac{1}{33^s} + \frac{1}{35^s} + \ldots \\ &\frac{-1}{30^s} - \frac{1}{42^s} - \frac{1}{66^s} - \frac{1}{70^s} - \frac{1}{78^s} - \frac{1}{102^s} - \frac{1}{105^s} - \frac{1}{110^s} - \frac{1}{114^s} - \ldots \end{aligned} \tag{5.4}$$

This strategy is continued infinitely many times (for all the combinations of four, all the combinations of 5, etc.) An interesting pattern begins to emerge when we arrange Equation 5.4 by magnitude:

$$1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{10^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{14^s} + \ldots \tag{5.5}$$

The pattern follows from the strategy we've used to create the sum. First note that all integers are either primes themselves or a product of unique primes (down to order) by the Fundamental Theorem of Arithmetic. This means that each term in Equation 5.5 is unique (an integer can't be the product of both m and n unique primes for $m \neq n$). In the sum, notice that any number which is a factor of a square prime is omitted from the denominators in the sum (e.g. 54 is a factor of $3^2 = 9$) since a given prime in a binomial of Euler's Inverted Product Formula is never repeated in a different binomial. Meanwhile, we see that any number which is a product of an odd amount of primes is prefixed by a negative in the denominator (e.g. $2 \cdot 3 \cdot 5 = -30$). Finally, any number which is a product of an even amount of primes is positive in the denominator (e.g. $2 \cdot 3 = 6$).

The coefficients to each term in Equation 5.5 form the Möbius Function. The function takes arguments n in the from $n = p_1 \cdot p_2 \cdots p_r$ where each p_i is a unique prime. It is:

Definition 5.1. The Möbius Function, $\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } \exists a \in \mathbb{P} : a^2 \mid n \\ (-1)^r & \text{if } \nexists a \in \mathbb{P} : a^2 \mid n \end{cases}$

5.2 Mertens Function and Consequences

With this in mind, we can write the Zeta Function in terms of the Möbius Function. Piecing together equations 5.1 and 5.5, along with Definition 5.1, we have:

Theorem 5.1. $\frac{1}{\zeta(s)} = \sum_n \frac{\mu(n)}{n^s}$.

The Möbius Function is important to the Riemann Hypothesis in so much as it's cumulative value, the Mertens Function, $M(k) = \sum_{n=1}^{k} \mu(n)$ opens another approach to solve the hypothesis. The Function is plotted below.

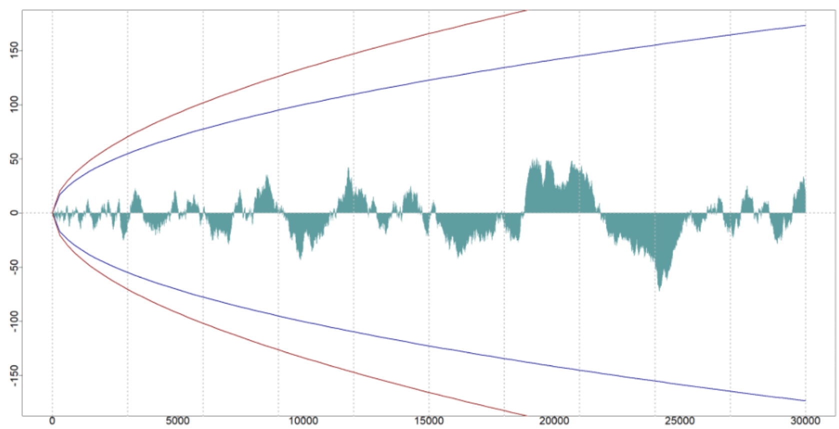

Figure 5.1: Mertens Function [10]

Stepping back, it is clear that the expected sum outcome of a fair binary (-1,1) event of n trials is 0. Of course, getting *exactly* 0 is unexpected for large n. Bernoulli was the first to show that the average sum is $\sqrt{n}$, which appears to be a bound for the Mertens Function (the blue line in Figure 5.1). In 1985, Odlyzko and Te Riele proved that the Mertens function was not bounded by $\sqrt{n}$ [11], a surprise given that the first counterexample appears no sooner than 10^{16}. A weaker statement, that the Mertens function is bounded by $n^{\frac{1}{2}+\epsilon}$ (the red line in Figure 5.1, where ϵ is epsilon, some arbitrarily small positive number) is precisely as strong as Riemann's Hypothesis.

6 J-Function

6.1 Definition and Möbius Inversion

The Prime Number Function from Section 2 is a step function. We now introduce another step function in an effort connect the Prime Counting Function and the Zeta Function.

$$\begin{aligned} J(x) &= \pi\left(x\right) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \frac{1}{4}\pi\left(x^{\frac{1}{4}}\right) + \ldots \\ &= \sum_{n} \sum_{p^n \leq x} \frac{1}{n} \end{aligned} \tag{6.1}$$

Notice that this is not an infinite sum since $\pi(y)$ is 0 whenever $y < 2$. The Prime Counting Function and J-Function are compared in Figure 6.1 below (the red line is the J-Function). While the Prime Counting Function and J-Function both increase by 1 whenever x is prime, the J-Function also increases by $\frac{1}{2}$ whenever $\sqrt{x}$ is prime, by $\frac{1}{3}$ whenever $\sqrt[3]{x}$ is prime, etc.

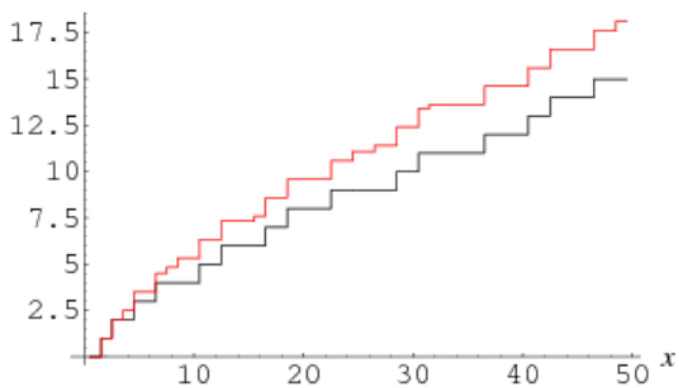

Figure 6.1: J-Function [12]

We've defined J in terms of π, and by the process of Mobius Inversion, can write π in terms of J, $\pi(x) = \sum_n \frac{\mu(n)}{n} J\left(x^{\frac{1}{n}}\right)$. We now try to express ζ in terms of J.

6.2 Writing Zeta in Terms of the J-Function

Recall that $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ from Section 4.1. Taking the logarithm, $\log(\zeta(s)) = \log\left(\frac{1}{1-\frac{1}{2^s}}\right) + \log\left(\frac{1}{1-\frac{1}{3^s}}\right) + \log\left(\frac{1}{1-\frac{1}{5^s}}\right) + \log\left(\frac{1}{1-\frac{1}{7^s}}\right) \dots$ which is $-\log\left(1-\frac{1}{2^s}\right) - \log\left(1-\frac{1}{3^s}\right) - \log\left(1-\frac{1}{5^s}\right) - \dots$ by the properties of logarithms. We are trying to connect the J-Function to the Zeta Function, but first need to prove a lemma.

Lemma 6.1. We would like a closed formula for the geometric series.

The sum of the first n terms is $S = \sum_{k=0}^{n-1} a \cdot r^k$.
Expanding the series, we have $S = a + ar + ar^2 + ... + ar^{n-1}$.
Multiplying by r, we then have $Sr = ar + ar^2 + ar^3 + ... + ar^n$.
Subtracting the two equations and grouping terms, we arrive at the solution:

$$S = \sum_{k=0}^{n-1} a \cdot r^k = a \cdot \left(\frac{1-r^n}{1-r}\right) \qquad \square$$

When $a = 1$ and $r \in (-1, 1)$, the geometric series simplifies to $\frac{1}{1-r}$. Taking the integral, $\int \frac{1}{1-r} = \int 1 + r + r^2 + r^3 + r^4 + ...$, which simplifies to $-\log(1-r) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...$ Since $0 < |\frac{1}{p^s}| < 1$, we can write each term in Euler's Product Formula as an infinite sum. We have:

$$\begin{gathered}
\log\left[\zeta(s)\right] = \\
\text{-}\log\left(1-\frac{1}{2^s}\right) - \log\left(1-\frac{1}{3^s}\right) - \log\left(1-\frac{1}{5^s}\right) - \cdots = \\
\left[\frac{1}{2^s} + \left(\frac{1}{2}\cdot\left(\frac{1}{2^s}\right)^2\right) + \left(\frac{1}{3}\cdot\left(\frac{1}{2^s}\right)^3\right) + \left(\frac{1}{4}\cdot\left(\frac{1}{2^s}\right)^4\right)\dots\right] + \\
\left[\frac{1}{3^s} + \left(\frac{1}{2}\cdot\left(\frac{1}{3^s}\right)^2\right) + \left(\frac{1}{3}\cdot\left(\frac{1}{3^s}\right)^3\right) + \left(\frac{1}{4}\cdot\left(\frac{1}{3^s}\right)^4\right)\dots\right] + \\
\left[\frac{1}{5^s} + \left(\frac{1}{2}\cdot\left(\frac{1}{5^s}\right)^2\right) + \left(\frac{1}{3}\cdot\left(\frac{1}{5^s}\right)^3\right) + \left(\frac{1}{4}\cdot\left(\frac{1}{5^s}\right)^4\right)\dots\right] + \\
\cdots \\
= \sum_p \sum_n \left(\frac{1}{n}\cdot\frac{1}{p^{ns}}\right)
\end{gathered} \tag{6.2}$$

Any term in this infinite sum of infinite sums can be written as an integral. Notice, $\frac{1}{n} \cdot \frac{1}{p^{ns}} = s \cdot \int_{p^n}^{\infty} \frac{1}{n} x^{-s-1}\, dx$ since $\int_{p^n}^{\infty} \frac{1}{n} x^{-s-1} dx = \frac{1}{n} \left(\frac{-1}{s} \cdot \frac{1}{x^s} \right) \Big|_{p^n}^{\infty} = \frac{1}{n} \left[(0) - \left(\frac{-1}{s} \cdot \frac{1}{p^{ns}} \right) \right] = \frac{1}{ns} \times \frac{1}{p^{ns}}$ which is $\frac{1}{s}$ multiples of $\frac{1}{n} \times \frac{1}{p^{ns}}$.

Rewriting Equation 6.2 with integrals and factoring out s, we have:

$$\begin{gathered} \frac{1}{s} \log\left[\zeta(s)\right] = \\ \left[\left(\int_{2}^{\infty} \frac{1}{1} \cdot x^{-s-1}\, dx \right) + \left(\int_{2^2}^{\infty} \frac{1}{2} \cdot x^{-s-1}\, dx \right) + \left(\int_{2^3}^{\infty} \frac{1}{3} \cdot x^{-s-1}\, dx \right) + \ldots \right] + \\ \left[\left(\int_{3}^{\infty} \frac{1}{1} \cdot x^{-s-1}\, dx \right) + \left(\int_{3^2}^{\infty} \frac{1}{2} \cdot x^{-s-1}\, dx \right) + \left(\int_{3^3}^{\infty} \frac{1}{3} \cdot x^{-s-1}\, dx \right) + \ldots \right] + \\ \left[\left(\int_{5}^{\infty} \frac{1}{1} \cdot x^{-s-1}\, dx \right) + \left(\int_{5^2}^{\infty} \frac{1}{2} \cdot x^{-s-1}\, dx \right) + \left(\int_{5^3}^{\infty} \frac{1}{3} \cdot x^{-s-1}\, dx \right) + \ldots \right] + \\ \cdots \\ = \sum_{p} \sum_{n} \left(\int_{p^n}^{\infty} \frac{1}{n} \cdot x^{-s-1}\, dx \right) \end{gathered} \tag{6.3}$$

Now consider the step function $g(x, p^n) = \left\{ \begin{array}{ll} 1 & \text{if } x \geq p^n \\ 0 & \text{if } x < p^n \end{array} \right\}$ [13]. See that $\displaystyle\int_{0}^{\infty} g(x, p^n)\, x^{-s-1}\, dx = \int_{0}^{p^n} g(x, p^n)\, x^{-s-1}\, dx + \int_{p^n}^{\infty} g(x, p^n)\, x^{-s-1}\, dx$. But this can simplify to $\displaystyle\int_{0}^{\infty} g(x, p^n)\, x^{-s-1}\, dx = \int_{p^n}^{\infty} x^{-s-1}\, dx$ since $g(x, p^n)$ is 0 whenever $0 \leq x \leq p^n$ and 1 whenever $x > p^n$.

Integrals are transparent to multiplying factors, so by subsituting the step function into Equation 6.3, we can write:

$$\frac{1}{s} \log\left[\zeta(s)\right] = \sum_{p^n \leq x} \sum_{n} \left(\int_{0}^{\infty} \frac{1}{n} g(x, p^n)\, x^{-s-1}\, dx \right) \tag{6.4}$$

The sum of the integrals is equal to the integral of the sums [14], so recognizing the definition of the J-Function in Equation 6.1, we can finally say that:

$$\frac{1}{s} \log\left[\zeta(s)\right] = \int_{0}^{\infty} J(x) x^{-s-1}\, dx \tag{6.5}$$

7 Explicit Formula

The main result of Riemann's 1859 paper was providing an explicit formula connecting the zero's of the Zeta Function with the distribution of prime numbers, seen below in Equation 7.1. His proof relied on the properties of Equation 6.5. We've already shown that the logarithmic integral $Li(x)$ is a decent approximation for the number of primes up to x, $\pi(x)$, but Equation 7.1 provides an exact formula connecting the distribution of the non-trivial zeros of the Zeta Function to the Prime Number Function via the J-Function. The truth of this formula was proved to be independent to the truth of Riemann's Hypothesis by von Mangoldt in 1895.

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{1}{t(t^2-1)\log t}\, dt \tag{7.1}$$

We already have a grasp on three of the four terms. The principle term is the aforementioned logarithmic integral defined in Section 3. The third term is a constant, equal to about 0.693. The fourth term can never be larger than 0.15. So the main focus of this section will be on the second term.

Examine each x^{ρ} in the second term, where ρ represents the non-trivial zeros of the zeta function. These zeros all occur on the critical strip (provided Riemann's Hypothesis is true), so each x^{ρ} can be written in the form $x^{\frac{1}{2}+bi}$ for real b. The magnitude of each of these x^{ρ}'s is then $\sqrt{x}$, so for a given x, the non-trivial zeros of the zeta function form a circle of radius $\sqrt{x}$ in the complex plane.

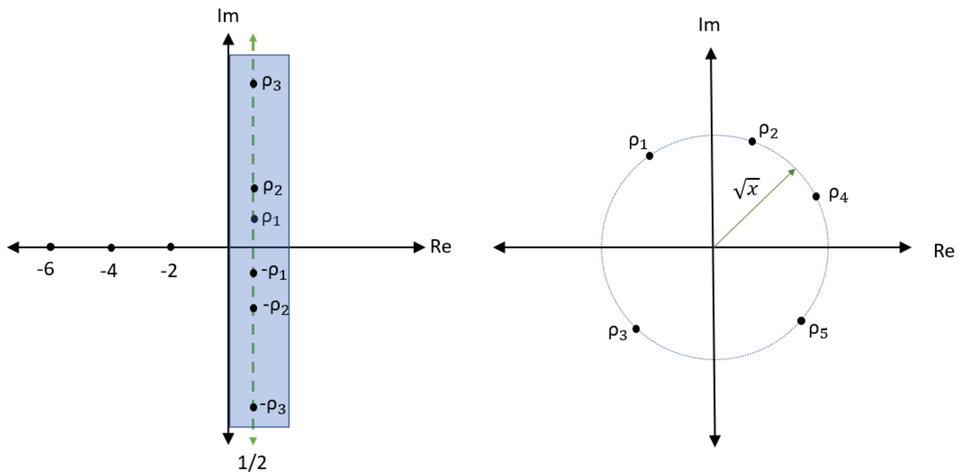

Figure 7.1: Critical Strip And x^ρ With Selected Zeros

Taking the logarithmic integral of each x^ρ we see the circle transformed to a counter-clockwise spiral, centering on πi. Since every non-trivial zero has a complex conjugate that is also a zero, there is also a clockwise spiral on the negative imaginary axis centered around $-\pi i$. The spirals get bigger as x gets bigger, and eventually the spiral in the positive imaginary axis and negative imaginary axis overlap (at about $x = 400$).

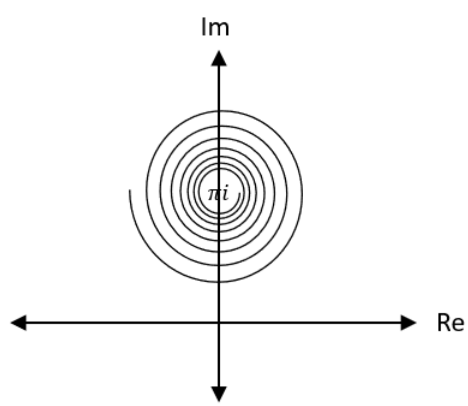

Figure 7.2: $Li(x^\rho)$

The final part of the second term is the summation. We already know each non-trivial zero has a complex conjugate, so for any zeta zero $\rho = a + bi$, the imaginary part cancels out ($\rho + \bar{\rho} = (a+bi) + (a-bi) = 2a$). Since both spirals are closing in on πi and $-\pi i$, the real parts of the summation should close in on 0.

Recall the Möbius Inversion in Section 6. To find the number of primes up to a number, say 1,000,000, we can say $\pi(1,000,000) = J(1,000,000) - \frac{1}{2}J(\sqrt{1,000,000}) - \frac{1}{3}J(\sqrt[3]{1,000,000}) + \ldots$. So to calculate the number of primes up to 1,000,000, we just need to execute Equation 7.1 for the N roots of 1,000,000 greater than 2. We can see the totality of this calculation (up to two digits) in Figure 7.3 below. See that the original approximation we had for $\pi(1,000,000)$, $Li(1,000,000) \approx 78627.55$, is only about 130 (-0.15%) less than the actual value of 79498.

N	$\frac{\mu(N)}{N} Li(x^{1/N})$	$\frac{-\mu(N)}{N} \sum_{\rho} Li(x^{\rho})$	$\frac{-\mu(N)}{N} \log 2$	$\frac{\mu(N)}{N} \int_{x^{1/N}}^{\infty} \frac{1}{t(t^2-1)\log t}\, dt$	Total
1	78627.55	-29.74	-0.69	0.00	78597.11
2	-88.80	0.11	0.34	0.00	-88.35
3	-10.04	0.30	0.23	0.00	-9.51
...	...	...	...	...	...
16	0.11	-0.00	-0.05	0.00	0.07
17	-0.08	-0.01	0.04	-0.01	-0.06
19	-0.06	-0.02	0.04	-0.01	-0.05
Total	78527.35	-29.37	0.04	-0.01	78498.00

Figure 7.3: Calculation of $\pi(1,000,000)$

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