Real Analysis Review

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1 Sets and Functions

1.1 Definitions

Definition 1.1. An element x is either a member of a **set** A (in which case we write $x \in A$) or it is not (in which case we write $x \notin A$). A set B is a **subset** of A, denoted $B \subseteq A$ if every element of B is an element of A. If there are elements of A that aren't in a subset B, then we call B a **proper subset** and can write $B \subset A$.

Definition 1.2. The **union** of sets A and B is denoted $A \cup B = \{x : x \in A or x \in B\}$. The **intersection** of sets A and B is denoted $A \cap B = \{x : x \in A and x \in B\}$. The **compliment** of A relative to B, denoted $A \setminus B$ is the set of all elements in A that aren't in B; $A \setminus B = \{x : x \in A and x \notin B\}$.

Definition 1.3. A function $f : A \to B$ maps elements from one set A, called the **domain**, to another set B, called the **codomain**. The set $f(A) \subseteq B$ is called the **range**. For a subset $S \subset A$, the set $f(S)$ is called the **image**. If $f : A \to B$ and $g : B \to C$ are two functions, then we can write the **composition of functions** as $g \circ f : A \to C$

Definition 1.4. A function $f : X \to Y$ is **injective** if distinct inputs have distinct outputs; that is if whenever $f(a) = f(b)$, it must be the case that $a = b$. It is **surjective** if every element in the codomain is mapped to by an element in the domain; that is if for all $y \in Y$, we can find a $x \in X$ such that $f(x) = y$. A function that is both injective and surjective is **bijective**.

Definition 1.5. The **cardinality** of a set S, denoted $|S|$ refers informally to it's size. A set can have zero elements (denoted), a finite number of elements, or an infinite number of elements. If there exists an injection between A and B, we can write $|A| \leq |B|$. A set is **denumerable** if there is a bijection from $\mathbb{N}$ to S (this indirectly says that $\mathbb{N}$ is the "smallest" infinity). A set is **countable** if it is either finite or denumerable. A set is **uncountable** if it is not countable.

1.2 Theorems

Theorem 1.1. Principle Of Induction: If $P(n)$ is a statement about $n \in \mathbb{N}$, then if $P(1)$ is true and if $P(k+1)$ being true is implied from $P(k)$ being true for any arbitrary k, then P is true for all n.

Theorem 1.2. Cantor, Schroder, Bernstein: if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection between A and B.

1.3 Examples And Problems

1) Prove $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$.

We try for dual containment. First assume $x \in A \cap (B \cup C)$. By the definition of intersect, x is a member of A. By the definition of union, x is also a member of B or C. So x must be a member of either $A \cap B$ or $A \cap C$; $x \in (A \cap B) \cup (A \cap C)$.

Next assume $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$ (so then $x \in B \cup C$). In the same way, if $x \in A \cap C$, then $x \in A$ and $x \in C$ (so then again $x \in B \cup C$). In either case, x is a member of A and $B \cup C$; $x \in A \cap (B \cup C)$.

5) Prove the integers are countably infinite.

We aim to show a bijection from $\mathbb{N}$ to $\mathbb{Z}$. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ where $f(x) = \begin{cases} \frac{x}{2}, & x \in 2\mathbb{N} \\ \frac{1-x}{2}, & x \notin 2\mathbb{N} \end{cases}$.

Let $y \in \mathbb{Z}$ be given. We look at three cases for y. If $y = 0$, then 1 in the domain maps to y since $\frac{-[1]+1}{2} = 0 = y$. If $y > 0$, then $2y$ in the domain maps to y since $\frac{[2y]}{2} = y$. Finally if $y < 0$, then $1 - 2y$ in the domain maps to y since $\frac{1-[1-2y]}{2} = y$. So f is surjective.

Next let $f(x_1), f(x_2) \in \mathbb{Z}$ with $f(x_1) = f(x_2)$ be given. Suppose $x_1 \neq x_2$. If x_1 and x_2 have opposite parity, then $f(x_1) \neq f(x_2)$ by the construction of f. If x_1 and x_2 are both even, then they can be written in the form $x_1 = 2m$ and $x_1 = 2n$ for distinct $m, n \in \mathbb{N}$. But then $\left(f(x_1) = \frac{2m}{2} = m\right) \neq \left(f(x_2) = \frac{2n}{2} = n\right)$. Finally if x_1 and x_2 are both odd, then they can be written in the form $x_1 = 2k - 1$ and $x_1 = 2l - 1$ for distinct $k, l \in \mathbb{N}$. But then $\left(f(x_1) = \frac{1-(2k-1)}{2} = 1 - k\right) \neq \left(f(x_2) = \frac{1-(2l-1)}{2} = 1 - l\right)$ and we have reached our contradiction. So $x_1 = x_2$ and f is injective and thus bijective.

2 Real Number System

2.1 Definitions

Definition 2.1. A **neighborhood** of a point x_0 in a space E is denoted $B_E(x_0, r) = \{x : x \in E, (x_0 - r) < x < (x_0 + r)\}$ for a positive distance r (for radius).

Definition 2.2. A set A is **bounded** if it has upper and lower bounds, that is if there exists values $u \in E$ and $l \in E$ such that for any $x \in A$, $l \leq x \leq u$

Definition 2.3. A value u is a **least upper bound** (supremum) of a set A, denoted $\sup\{A\}$, if it is both an upper bound of A, and is less than or equal to any other upper bound of A. Likewise a value l is a **greatest lower bound** (infimum) of a set A, denoted $\inf\{A\}$, if it is both a lower bound of A, and is greater than or equal to any other lower bound of A.

Definition 2.4. A set F equipped with two closed, binary, associative, commutative, and distributive operations whose member elements all have inverses is an algebraic structure called a **field**. Where F is the set and $+$ and $\cdot$ are the operations, we write $(F, +, \cdot)$.

Definition 2.5. A set S is **ordered** if we can establish a relation (reflexive, symmetric, and transitive) for all elements $x, y \in S$; For a given x, either $x < y$, $x = y$, or $x > y$. This condition is called **trichotomy**.

Definition 2.6. A set S is **complete** if every nonempty subset of S that has an upper bound also has a least upper bound. The real numbers are the only complete totally ordered field (down to isomorphism).

Definition 2.7. An **open interval** of values a, b is the set $(a, b) = \{x : a < x < b\}$. Similarly a **closed interval** is the set $[a, b] = \{x : a \leq x \leq b\}$. Half open or half closed intervals are defined as expected. A sequence of intervals I_n with $n \in \mathbb{N}$ is **nested** if we have $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ for all n.

2.2 Theorems

Theorem 2.1. Order Of Real Numbers: The Real Numbers can be partitioned into three sets: $\mathbb{R}_+$, $\{0\}$, and $\mathbb{R}_-$. Familiar properties of $\mathbb{R}$ hold. For example, if $a, b \in \mathbb{R}_+$ with $a > b$, then for $c \in \mathbb{R}_-$ then $ac < bc$.

Theorem 2.2. Arbitrarily Close Points Implies Equality: If an element a has the property that $0 \leq a < \varepsilon$ for any arbitrary $\varepsilon > 0$, then $a = 0$.

Theorem 2.3. Triangle Inequality: If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$. Two immediate corollaries are that $|a - b| \leq |a| + |b|$ and $||a| - |b|| \leq |a| + |b|$.

Theorem 2.4. Least Upper Bound Property: A value u is the least upper bound of a set S if and only if for any value $v < u$, we can find a value $s \in S$ such that $v < s < u$. Formally, $u = \sup\{S\}$ if and only if for all $\varepsilon > 0$, there exists a $s \in S$ such that $u - \varepsilon < s$.

Theorem 2.5. Archimedean Property: If $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$, then there exists a $n \in \mathbb{N}$ such that $nx > y$.

Theorem 2.6. $\mathbb{Q}$ is dense in $\mathbb{R}$: Between any two real numbers, there is a rational number (and an irrational number).

Theorem 2.7. If $I_n = [an, bn]$ for $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then there is a real number r such that $r \in I_n$ for any n. If $\inf\{bn - an : n \in \mathbb{N}\}$, then this r is unique.

Theorem 2.8. The Real Numbers aren't countable.

2.3 Examples and Problems

1) If $a, b \in \mathbb{R}$, prove the following:

a. If $a + b = 0$, then $b = -a$.

By left addition, we have $(-a) + a + b = (-a) + 0$. By the property of zero and grouping, we then have $(-a + a) + b = (-a)$. By the properties of additive inverse, we see $0 + b = -a$ and finally $b = -a$

b. $-(-a) = a$.

By left addition, we have $(-a) - (-a) = (-a) + a$. By cancellation and the properties of additive inverses, we see $0 = 0$

c. $(-1)a = -a$.

We first prove a lemma. **Lemma 1:** For all field elements, $0a = 0$. Start with $1a = 1a$. So $(0+1)a = 1a$ by the properties of zero. By the distributive property of fields, $0a + 1a = 1a$. Then by right cancellation, $0a = 0$.

We know that $(-a)$ is the unique field element such that $(-a)+a = 0$. So if we can show $(-1)a + a = 0$, we will have shown $(-a) = (-1)a$. Start with the conclusion of Lemma 1, $0a = 0$. By the properties of additive inverses, $(-1 + 1)a = 0$. By the distributive properties of fields, $(-1)a + 1a = 0$. By the multiplicative identity, $(-1)a + a = 0$ as required. So $(-1)a = -a$

d. $(-1)(-1) = 1$.

From question 1c, we see $(-1)(-1) = -(-1)$. From question 1b, we see that $-(-1) = 1$ as required. There are two immediate corollaries.

Corollary 1: For all $a, b \in \mathbb{R}_+$, $(-a)(-b) = ab$. See that $(-a)(-b) = -1(a) - 1(b)$ (from question 1c) and $-1(a) - 1(b) = (-1)(-1)ab = 1ab = ab$ by commutativity.

Corollary 2: $ab \in \mathbb{R}_-$ if and only if $a \in \mathbb{R}_+$ while $b \in \mathbb{R}_-$ (or vise versa). By the order properties of the real numbers, if $a, b \in \mathbb{R}_+$, then $ab \in \mathbb{R}_+$. By Corollary 1, if $a, b \in \mathbb{R}_-$, then $ab \in \mathbb{R}_+$ as well. By Lemma 1, if either $a = 0$ or $b = 0$, then $ab = 0$. The only remaining case is if a and b have opposite signs. Since field elements $f \in \mathbb{R}_-$ exist (if $f \in \mathbb{R}_-$, take $-f$), and since fields are closed, it must be the case that whenever $a \in \mathbb{R}_+$ and $b \in \mathbb{R}_-$, $ab \in \mathbb{R}_-$.

6) Show that there does not exist a rational s such that $s^2 = 6$.

We try for the general case but first prove a Lemma. **Lemma 2:** If p and q are coprime, then any power of p, say p^k, and q are also coprime. By the Fundamental Theorem of Arithmetic, p and q have unique prime decompositions, say $p = N_1 \cdot N_2 \cdots N_t$ and $q = M_1 \cdot M_2 \cdots M_s$. Then any power k of p can be written $p^k = N_1^k \cdot N_2^k \cdots N_t^k$. By the definition of coprime, $M_i \nmid N_j$ for any choice of i or j. It must then be the case that $M_i \nmid N_j^k$.

Now consider an integer n whose square root is rational, $\sqrt{n} = \frac{p}{q}$ (where p and q are coprime). Then $n = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$ and so $nq^2 = p^2$. Clearly $q|p^2$. But since q and p are coprime, by Lemma 2, q and p^2 are coprime also. So $q = 1$.

We can then write $n = p^2$. This implies that the only perfect squares have rational roots. Since 6 is not a perfect square, we have shown the desired result.

4) Show that $|x - a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$

Assume $|x - a| < \varepsilon$. By the definition of absolute value and trichotomy, both $(x - a) < \varepsilon$ and $-(x - a) < \varepsilon$. In the first inequality, we see $x < \varepsilon + a$ by right addition and commutativity. In the second inequality, we see $(x - a) > -\varepsilon$ and then $x > a - \varepsilon$ again by right addition and commutativity. Combining these inequalities, we see $a - \varepsilon < x < a + \varepsilon$ as required.

Next assume $a - \varepsilon < x < a + \varepsilon$. In the right-most inequality, we have $x < a + \varepsilon$ so $x - a < \varepsilon$ by the properties of additive inverses, left addition, and commutativity. In the left-most inequality, we have $a - \varepsilon < x$ so that $-(a - \varepsilon) > -x$ (multiply by -1), and $\varepsilon > -x + a$ (by distribution and left addition), and finally $\varepsilon > -(x - a)$ (by redistribution). This proves $|x - a| < \varepsilon$.

2) If $S = \{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

This example shows that the supremum and infimum are not operation preserving. We claim 1 is the least upper bound and -1 is the greatest lower bound.

First note that the supremum of the sequence $\frac{1}{n}$ for natural number n is 1 since the series is monotonically decreasing from 1 (1 is an upper bound), and since 1 is a member of the set (so is the least upper bound). Next, we show that the limit of this sequence is 0 to show it's infimum is 0. Let $\varepsilon > 0$ be given and consider $N = \frac{2}{\varepsilon}$ (if N is not a member of the natural numbers, the Archimedean Property guarantees that we can find a larger value that is a natural number). Then we see $d\left(\frac{1}{N}, 0\right) = d\left(\frac{1}{\frac{2}{\varepsilon}}, 0\right) = \frac{\varepsilon}{2} < \varepsilon$ as required.

Since S subtracts these sets, it is clear that $\sup\{S\} = \sup\{\frac{1}{n}\} - \inf\{\frac{1}{m}\} = 1 - 0 = 1$. Similarly, the infimum is exactly the number whose value minimizes the first term and maximizes the second term; $\inf\{S\} = \inf\{\frac{1}{n}\} - \sup\{\frac{1}{m}\} = 0 - 1 = -1$.

4) Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$ and let $aS = \{as : s \in S\}$. What is $\sup(aS)$ and $\inf(aS)$? How does that change if $a < 0$?

Since S is nonempty and bounded, S has both a least upper bound and a greatest lower bound by the completeness of $\mathbb{R}$. Let $s \in S$ be given. Then $\inf\{S\} \leq s \leq \sup\{S\}$ by the definition of upper and lower bounds. Consequently, $a \cdot \inf\{S\} \leq a \cdot s \leq a \cdot \sup\{S\}$ since $a \in \mathbb{R}_+$. So $a \cdot \sup\{S\}$ is an upper bound and $a \cdot \inf\{S\}$ is a lower bound of the set aS.

Since aS is a non-empty and bounded subset of the Real Numbers, it has a least upper bound and greatest lower bound. Then $\inf\{aS\} \leq a \cdot s \leq \sup\{aS\}$ by the definition of boundedness. Consequently, we see $\frac{1}{a}\inf\{aS\} \leq s \leq \frac{1}{a}\sup\{aS\}$. So $\frac{1}{a}\sup\{aS\}$ is an upper bound and $\frac{1}{a}\inf\{aS\}$ is a lower bound of S. Since $\sup\{S\}$ is the *least* upper bound of S, $\sup\{S\} \leq \frac{1}{a}\sup\{aS\}$. Likewise, since $\inf\{S\}$ is the *greatest* lower bound, $\frac{1}{a}\inf\{aS\} \leq \inf\{S\}$. But then $a \cdot \sup\{S\} \leq \sup\{aS\}$ and $\inf\{aS\} \leq a \cdot \inf\{S\}$ and we have shown that $\sup\{aS\} = a \cdot \sup\{S\}$ and $\inf\{aS\} = a \cdot \inf\{S\}$ as required.

If instead $a < 0$, then $\inf\{S\} \leq s \leq \sup\{S\}$ implies $a \cdot \inf\{S\} \geq a \cdot s \geq a \cdot \sup\{S\}$. So we have $a \cdot \inf\{S\}$ as an upper bound and $a \cdot \sup\{S\}$ as a lower bound of the set aS.

Since aS is closed and bounded, it has least upper bounds and greatest lower bounds; $\inf\{aS\} \leq as \leq \sup\{aS\}$. Then $\frac{1}{a}\inf\{aS\} \geq s \geq \frac{1}{a}\sup\{aS\}$ and we have additional bounds for S. But recall that $\sup\{S\}$ is the *least* upper bound and that $\inf\{S\}$ is the *greatest* lower bound of S, so $\inf\{S\} \leq \frac{1}{a}\sup\{aS\} \leq s \leq \frac{1}{a}\inf\{aS\} \leq \sup\{S\}$. Then $a\inf\{S\} \geq \sup\{aS\} \geq as \geq \inf\{aS\} \geq a\sup\{S\}$ and we have shown $a\inf\{S\}$ is the supremum and $a\sup\{S\}$ is the infimum.

3 Series and Sequences

3.1 Definitions

Definition 3.1. A **sequence** $\{x_n\}$ of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. We identity each **term** of x by it's index. The **m-tail** of a sequence are all the values of the sequence excluding the first m terms.

Definition 3.2. A sequence $\{x_n\}$ **converges** to a limit L if for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that whenever $n > N$, it must be the case that $d(x_n, L) < \varepsilon$. If a sequence doesn't converge, it is said to be **divergent**. If a sequence tends to $\pm\infty$, we say it is **properly divergent**.

Definition 3.3. A sequence $\{x_n\}$ is **monotone** if it is either increasing or decreasing; that is if either $x_1 \leq x_2 \leq \cdots \leq x_n$ or $x_1 \geq x_2 \geq \cdots \geq x_n$ for all $n \in \mathbb{N}$.

Definition 3.4. A **subsequence** of a sequence $\{x_n\}$ is any subset of the sequence whose index values are strictly increasing, i.e. if $n_1 < n_2 < ...$, then $x_{n_1}, x_{n_2}, \ldots$ is a subsequence.

Definition 3.5. The **limit superior** of a sequence $\{x_n\}$, denoted $\limsup\{x_n\}$, is the infimum of the set of $V = \{v : v < x_n\}$ for at most finite n.

Definition 3.6. A sequence $\{x_n\}$ is **Cauchy** if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that whenever $m, n > N$, it must be the case that $d(x_m, x_n) < \varepsilon$.

Definition 3.7. A sequence of real numbers $\{x_n\}$ is **contractive** if there is a constant $C \in (0, 1)$ such that $|x_{n+2} - x_{n+1}| \leq C\, |x_{n+1} - x_n|$.

Definition 3.8. For a sequence $X = \{x_n\}$, the **infinite series** generated by X is the sequence defined by $\sum\limits_{n=1}^{\infty} x_n$.

Definition 3.9. A series $\sum x_n$ is convergent if and only if for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that whenever $m > n > N$, it must be the case that $|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon$.

3.2 Theorems

Theorem 3.1. The limit is unique.

Theorem 3.2. If a sequence has a limit, it is bounded.

Theorem 3.3. The tail of a sequence converges if and only if the sequences converges to the same limit.

Theorem 3.4. Arithmetic of limits is as expected. That is, for real number r and sequences $\{x_n\}$ and $\{y_n\}$ that converge to X and Y respectively, $\lim\limits_{n\to\infty}\{x_n + y_n\} = X + Y$, $\lim\limits_{n\to\infty}\{x_n \cdot y_n\} = X \cdot Y$, $\lim\limits_{n\to\infty}\{r \cdot x_n\} = rX$, and $\lim\limits_{n\to\infty}\{|x_n|\} = |X|$.

Theorem 3.5. If the limit of the ratio of consecutive terms in a sequence of positive real numbers converges to a value less than 1, the limit of the sequence is 0.

Theorem 3.6. A monotone sequence is convergent if and only if it is bounded.

Theorem 3.7. If a sequence $\{x_n\}$ converges to a value L, then any subsequence of $\{x_n\}$ also converges to L. This implies that any sequence with two subsequences that do not converge to the same limit is divergent.

Theorem 3.8. Any sequence has a monotonic subsequence.

Theorem 3.9. Bolzano-Weierstrass Theorem: Any bounded sequence has a convergent subsequence.

Theorem 3.10. A sequence is convergent if and only if it is Cauchy.

Theorem 3.11. Every contractive sequence is Cauchy.

Theorem 3.12. If $\sum x_n$ converges, then $\lim\limits_{n\to\infty}\{x_n\} = 0$

3.3 Examples and Problems

16) Show that $\lim_{n\to\infty}\left(\frac{n^2}{n!}\right) = 0$.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x \geq (3 + \frac{1}{\varepsilon}), x \in \mathbb{N}\}$. Then whenever $(n \in \mathbb{N}) > N$, it must be the case that $|\frac{n^2}{n!} - 0| = \frac{n}{(n-1)!} = \frac{n}{(1)\cdot(2)\cdot(3)\cdots(n-2)(n-1)} < \frac{n}{(n-2)(n-1)} = \frac{n}{n^2-3n+2} = \frac{n}{n(n-3+\frac{2}{n})} < \frac{1}{n-3} < \frac{1}{N-3} \leq \frac{1}{[3+\frac{1}{\varepsilon}]-3} = \frac{1}{1/\varepsilon} = \varepsilon$ as required.

5) Use the definition of the limit of a sequence to establish the following limits.

a. $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = 0$.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x \geq \frac{1}{\varepsilon}, x \in \mathbb{N}\}$. Then whenever $(n \in \mathbb{N}) > N$, it must be the case that $|\frac{n}{n^2+1} - 0| < \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N} \leq \frac{1}{[1/\varepsilon]} = \varepsilon$ as required.

b. $\lim_{n\to\infty}\left(\frac{2n}{n+1}\right) = 2$.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x \geq \frac{1}{\varepsilon}, x \in \mathbb{N}\}$. Then whenever $(n \in \mathbb{N}) > N$, it must be the case that $|\frac{2n}{n+1} - 2| = \frac{2}{n+1} < \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} \leq \frac{1}{[1/\varepsilon]} = \varepsilon$ as required.

c. $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x \geq \frac{13}{4\varepsilon}, x \in \mathbb{N}\}$. Then whenever $(n \in \mathbb{N}) > N$, it must be the case that $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{3n+1}{2n+5} - \frac{3n+\frac{15}{2}}{2n+5}\right| = \left|\frac{\frac{-13}{2}}{2n+5}\right| = \frac{13}{4n+10} < \frac{13}{4n} < \frac{13}{4N} \leq \frac{13}{4[\frac{13}{4\varepsilon}]} = \varepsilon$ as required.

d. $\lim_{n\to\infty}\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x \geq \frac{5}{4\varepsilon}, x \in \mathbb{N}\}$. Then whenever $(n \in \mathbb{N}) > N$, it must be the case that $\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{n^2-1}{2n^2+3} - \frac{n^2+\frac{3}{2}}{2n^2+3}\right| = \left|\frac{-5/2}{2n^2+3}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2} < \frac{5}{4n} < \frac{5}{4N} \leq \frac{5}{4[\frac{5}{4\varepsilon}]} = \varepsilon$ as required.

2) Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Show x_n is bounded and monotone. Find the limit.

The sequence is monotonically decreasing. We proceed by induction. The base case is that $x_2 < x_1$. Since $x_1 > 1$, $(x_1 - 1)(x_1 - 1) > 0$. Then $x_1^2 - 2x_1 + 1 > 0 \implies x_1^2 + 1 > 2x_1 \implies x_1 + \frac{1}{x_1} > 2 \implies x_1 - (2 - \frac{1}{x_1}) > 0$ and we have $x_1 - x_2 > 0$ as required.

The sequence is bounded below by 1. We argue by induction. The base case is clear– since $x_1 > 1$, $0 < \frac{1}{x_n} < 1$ and so $x_2 = 2 - \frac{1}{x_n} > 2 - [1] = 1$. The inductive hypothesis is that $x_k > 1$ for any k. See that $x_{k+1} = x_k + \frac{1}{x_k} > 1 + \frac{1}{x_k} > 1$ and we have proven the sequence is bounded.

Since the sequence is bounded and monotonic, it has a limit, call it L. Sequences converge to a limit if and only if their tail converges to the same limit. So $L = \lim_{n\to\infty}\{x_n\} = \lim_{n\to\infty}\{x_{n+1}\} = \lim_{n\to\infty}\{2 - \frac{1}{x_n}\}$. By the properties of limits, we see $L = 2 - \lim_{n\to\infty}\{\frac{1}{x_n}\} = 2 - \frac{1}{L}$. So $L + \frac{1}{L} = 2$ and we see the limit is 1.

7) Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$. Determine whether x_n converges or diverges.

The sequence diverges. First, see that the sequence is monotonically increasing– whenever $x_1 > 1$, $x_{n+1} = x_n + \frac{1}{x_n} > x_n$.

Next, see that the sequence is unbounded. We argue by contradiction. If the sequence had an upper bound, it would have a limit, call it L. But then $L = \lim_{n\to\infty}\{x_{n+1}\} = \lim_{n\to\infty}\{x_n + \frac{1}{x_n}\} = \lim_{n\to\infty}\{x_n\} + \lim_{n\to\infty}\{\frac{1}{x_n}\} = L + \frac{1}{L}$, which is impossible since there is no L whereby $0 = \frac{1}{L}$.

4) Show that $\{1-(-1)^n+\frac{1}{n}\}_{n=1}^{\infty}$ and $\{\sin(\frac{n\pi}{4})\}_{n=1}^{\infty}$ are divergent.

In the first case, we show two subsequences that converge to different limits. Consider the subsequence of every even term, $s_{n_1}=\frac{1}{n}$ and the subsequence of every odd term, $s_{n_2}=2+\frac{1}{n}$. We have previously shown that the limit of the sum is the sum of the limit, and that $\lim_{n\to\infty}\{\frac{1}{n}\}=0$, so see that the first sequence converges to 0, and the second sequence converges to 2.

In the second case, we can show that the distance between any adjascent terms cannot be made arbitrarily small. More formally, we recall that a sequence is convergent if and only if it is Cauchy, and to prove the sequence is not Cauchy, we can choose an $\varepsilon>0$ such that for any choice of $N\in\mathbb{N}$, we can always find values $m,n>N$ such that $|x_m-x_n|\geq\varepsilon$.

So consider $\varepsilon=1-\frac{\sqrt{2}}{2}$ and let $N\in\mathbb{N}$ be given. Since the sine function is cyclic, the sequence $\{\sin(\frac{n\pi}{4})\}_{n=1}^{\infty}$ exclusively takes values $\{0,\pm\frac{\sqrt{2}}{2},\pm1\}$. Then for any choice of n, we can choose $m=n+1$ and it must be the case that the distance between m and n is greater than or equal to ε. So the sequence is divergent.

5) Prove that the "shuffled" sequence $Z=\begin{cases}x_n, & \text{if } n \text{ is odd}\\ y_n, & \text{if } n \text{ is even}\end{cases}$ is convergent if and only if $X=\{x_n\}_{n=1}^{\infty}$ and $Y\{y_n\}_{n=1}^{\infty}$ converge to the same limit.

Assume the shuffled sequence converges to a limit L. Let $\varepsilon>0$ be given. Then by definition, there exists a $N\in\mathbb{N}$ such that $|z_n-L|<\varepsilon$. But z_n is either $x_{(n+1)/2}$ or $y_{n/2}$. So clearly the same choice of N shows that the limit of X and Y are both L.

Next assume that $\lim_{n\to\infty}\{x_n\}=\lim_{n\to\infty}\{y_n\}=L$. So there exists values $N,M\in\mathbb{N}$ such that whenever $n>N$, $|x_n-L|<\varepsilon$ and whenever $n>M$, $|y_n-L|<\varepsilon$. Choosing the larger of these two values shows that $|z_n-L|<\varepsilon$

4) Show that if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy, then $\{x_n+y_n\}_{n=1}^{\infty}$ and $\{x_n \cdot y_n\}_{n=1}^{\infty}$ are also Cauchy.

Let $\varepsilon > 0$ be given. By the definition of Cauchy, there exists a $N \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\varepsilon}{2}$ and $|y_n - y_m| < \frac{\varepsilon}{2}$ whenever $m, n > N$.

Then $|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ as required and we have shown the sum of Cauchy sequences is also Cauchy.

We can similarly show that the product of Cauchy sequences are Cauchy. Since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy, they have limits. Since they have limits, they have upper bounds, call them U_1 and U_2. Let $U = max\{U_1, U_2\}$. Then by the definition of Cauchy, we can find a $M \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\varepsilon}{2U}$ and $|y_n - y_m| < \frac{\varepsilon}{2U}$ whenever $m, n > M$.

See that whenever $m, n > M$, it must be the case that:

$$\begin{aligned}
|(x_n y_n) - (x_m y_m)| &= \\
|(x_n y_n) - (x_n y_m - x_n y_m) - (x_m y_m)| &= \quad \text{add zero} \\
|x_n(y_n - y_m) + y_m(x_n - x_m)| &\leq \quad \text{regroup and pull out common factor} \\
|x_n||y_n - y_m| + |y_m||x_n - x_m| &< \quad \text{triangle inequality} \\
[U][\frac{\varepsilon}{2U}] + [U][\frac{\varepsilon}{2U}] = \varepsilon & \quad \text{as required}
\end{aligned}$$

and we have shown the product is Cauchy.

2) Show that $\{\frac{n+1}{n}\}_{n=1}^{\infty}$ and $\{\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\}_{n=1}^{\infty}$ are Cauchy.

Let $\varepsilon > 0$ be given. Consider $N = \inf\{x : x > \frac{2}{\varepsilon}, x \in \mathbb{N}\}$. Then whenever $m, n > N$, it must be the case that $|\frac{n+1}{n} - \frac{m+1}{m}| = |(1+\frac{1}{n}) - (1+\frac{1}{m})| = |\frac{1}{n} - \frac{1}{m}| \le |\frac{1}{n}| + |\frac{1}{m}| < \frac{1}{N} + \frac{1}{N} = \frac{1}{2/\varepsilon} + \frac{1}{2/\varepsilon} = \varepsilon$ as required and we've shown $\{\frac{n+1}{n}\}$ is Cauchy.

Now consider $M = \inf\{x : x > \log_2(1/\varepsilon) + 1, x \in \mathbb{N}\}$. Without loss of generality, assume $m > n$. Then whenever $m, n > M$, it must be the case that $\left|\sum_{n=1}^{\infty} \frac{1}{m!} - \sum_{m=1}^{\infty} \frac{1}{n!}\right| = \left|\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}\right|$.

Before proceeding, we prove two lemmas:

Lemma 1: $2^n \le (n+1)!$ for all natural number n.

We use induction on n. The base case is obvious– $2^{[1]} = 2 \le ([1]+1)! = 2$. Now assume $2^k \le (k+1)!$ for some k. We'd like to show that $2^{k+1} \le (k+2)!$, and have $2^{k+1} = 2 \cdot 2^k \le 2 \cdot (k+1)!$ by the inductive hypothesis. Since $2 \le (k+2)$, we finally have $2^{k+1} \le (k+2) \cdot (k+1)! = (k+2)!$ as required.

Lemma 2: $\frac{1}{2^{n-1}} > \sum_{i=n}^{\infty} \frac{1}{2^i}$ for all natural n.

If k is larger than $n+2$ (the first two cases can be computed directly), we can transform all but one of the fractions to have the same denominator. Rewriting the inequality this way, we have $\frac{2^{k-2}+2}{2^{k-1}} > (\frac{2^{k-2}+1}{2^{k-1}}) + \frac{1}{2^k}$ since the rightmost term is less than $\frac{1}{2^{k-1}}$.

Using the above lemmas, we can continue with our proof. We have:

$$\begin{aligned}
&\left|\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!})\right| \le \\
&\left|\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}\right| < && \text{From Lemma 1} \\
&\frac{1}{2^{n-1}} < \frac{1}{2^{M-1}} \le && \text{From Lemma 2 and recalling } n > M \\
&\frac{1}{2^{[\log_2(1/\varepsilon)+1]-1}} = \frac{1}{2^{\log_2(1/\varepsilon)}} = \varepsilon && \text{Via subsititution}
\end{aligned}$$

and so have shown the sequence is Cauchy.

9) Show that $\sum_{n=1}^{\infty} \cos n$ is divergent while $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

Imagine that $\sum_{n=1}^{\infty} \cos n$ was convergent, so that we would have $\lim_{n\to\infty} \{\cos(n)\} = 0$. Then given a value, say $\frac{1}{3}$, we would be able to find a natural number N such that $|\cos(n) - 0| < \frac{1}{3}$ whenever $n > N$. But see a choice of $2n > n$ gives $|\cos(2n)| = |2 \cdot \cos^2(n) - 1|$ by the double angle formula for cosine, and that $|2 \cdot \cos^2(n) - 1| \leq \left|2 \cdot \frac{1}{9} - 1\right| = \frac{7}{9} > \frac{1}{3}$, a contradiction.

The second series converges by the comparison test. Since $\cos(n) \in [-1, 1]$ we see $\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \leq \sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right|$. Clearly the partial sums are monotone, so if we can show they are bounded, we will have proven the series is convergent by the monotone convergence theorem. To show the sequence of partial sums is bounded, it suffices to show a subsequence of the partial sums is bounded.

Consider the subsequence whose index is determined by $k_i = 2^i - 1$, $s_{k_i} = \{s_1, s_3, s_7, s_{15}, s_{31}, \dots\}$. So the terms of the sequence are:

$$s_{k_1} = s_1 = \frac{1}{1^2} = 1$$
$$s_{k_2} = s_3 = s_1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) < 1 + \left(\frac{2}{2^2}\right) \leq 1 + \left(\frac{1}{2}\right)$$
$$s_{k_3} = s_7 = s_3 + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) < s_3 + \left(\frac{4}{4^2}\right) \leq 1 + \frac{1}{2} + \left(\frac{1}{2^2}\right)$$
$$\vdots$$

Then for any k_j we see $0 < s_{k_j} < \sum_{n=0}^{j} (\frac{1}{2})^2$. On the right, we have previously shown that geometric series in the form ar^n (where $0 < r < 1$) converge to $\frac{a}{r-r}$. So the subsequence is bounded as required.

3) By using partial fractions, show $\sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$:

We'd first like to find the a partial sum decomposition of $\frac{1}{n(n+1)(n+2)}$, so want to find values A, B, C such that $\frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} = \frac{1}{n(n+1)(n+2)}$. See that we have:

$$\begin{aligned} A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1) &= 1 && \text{multiplying by largest term} \\ An^2 + 3An + 2A + Bn^2 + 2Bn + Cn^2 + Cn &= 1 && \text{expanding} \\ (C+B+A)\,n^2 + (C+2B+3A)\,n + 2A &= 1 && \text{grouping terms} \end{aligned}$$

Further, see that the coefficients to each term with a factor of n must be zero (and so the term without n must be 1). This gives us a set of three equations and three unknowns. We can solve this system of linear equations using techniques from linear algebra. We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Proceeding with backward substitution, we have $A = \frac{1}{2}$, $B = -1$, and $C = \frac{1}{2}$, so see $\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4} = \frac{1}{n(n+1)(n+2)}$. We can then write the terms of the sequence as follows:

$$\begin{aligned}
S_1 &= \frac{1}{1\cdot 2\cdot 3} = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \left(\frac{1}{6}\right) \\
S_2 &= \frac{1}{2\cdot 3\cdot 4} = \frac{1}{4} - \frac{1}{3} + \left(\frac{1}{8}\right) \\
S_3 &= \frac{1}{3\cdot 4\cdot 5} = \left(\frac{1}{6}\right) - \frac{1}{4} + \frac{1}{10} \\
S_4 &= \frac{1}{4\cdot 5\cdot 6} = \left(\frac{1}{8}\right) - \frac{1}{5} + \frac{1}{12} \\
&\vdots
\end{aligned}$$

Combining fractions with the same denominator and then grouping by plus and minus, we are left with the below sum.

$$\frac{1}{4} + \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots\right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots\right)$$

This proves the series sums to $\frac{1}{4}$

4 Limits

4.1 Definitions

Definition 4.1. A point $c \in \mathbb{R}$ is a **cluster point** of $A \subseteq \mathbb{R}$ provided there is always a point in A that is within an arbitrary neighborhood of c; that is, given any $\varepsilon > 0$, one can find a point $x \neq c$ in A such that $|c - x| < \varepsilon$.

Definition 4.2. A function $f : A \to \mathbb{R}$ has a **limit** L at a cluster point c, denoted $\lim\limits_{x \to c} f(x) = L$, if for any arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, it must be the case that $|f(x) - L| < \varepsilon$.

Definition 4.3. A function $f : A \to \mathbb{R}$ **does not have a limit** at a cluster point c if there is a sequence $\{x_n\}$ that converges to c (with each $x_i \neq c$) but the sequence $\{f(x_n)\}$ does not converge.

Definition 4.4. A function $f : A \to \mathbb{R}$ has a **right-hand limit** at c, denoted $\lim\limits_{x \to\ x+} f(x) = L$ if c is a cluster point of the set $\{x \in A : x > c\}$ and if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < x - c < \delta$, $|f(x) - L| < \varepsilon$. Analogous definitions follow for the left-hand limit (with c a cluster point of $\{x \in A : x < c\}$, and the condition $0 < c - x < \delta$).

Definition 4.5. A function $f : A \to \mathbb{R}$ **tends to infinity** at a cluster point c, denoted $\lim\limits_{x \to c} f(x) = \infty$, if for any $\alpha \in \mathbb{R}$ there exists a $\delta > 0$ such that whenever $0 < x - c < \delta$, $f(x) > \alpha$.

Definition 4.6. A function $f : A \to \mathbb{R}$ has a **limit at infinity**, denoted $\lim\limits_{x \to \infty} f(x) = L$, if $(\alpha, \infty) \subseteq A$ for some $\alpha \in \mathbb{R}$, and if given any $\varepsilon > 0$, there exists a $\delta > \alpha$ such that whenever $x > \delta$, $|f(x) - L| < \varepsilon$.

Definition 4.7. A function $f : A \to \mathbb{R}$ **tends to infinity as x approaches infinity**, denoted $\lim\limits_{x \to \infty} f(x) = \infty$, if when $(a, \infty) \subseteq A$ and when given a $\alpha > 0$, there exists a $\delta > \alpha$ such that whenever $x > \delta$, $f(x) > \alpha$.

Definition 4.8. A sequence of functions $f_n : A \to \mathbb{R}$ converges **pointwise** to the function $f(x)$ provided for all $\varepsilon > 0$ and for all $x \in A$, there exists a $N \in \mathbb{N}$ such that whenever $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

The sequence of functions converges **uniformly** to f provided for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that whenever $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in A$.

4.2 Theorems

Theorem 4.1. The limit is unique.

Theorem 4.2. If a function has a limit at c, it is bounded on a neighborhood of c.

Theorem 4.3. The **arithmetic of limits** is as expected; if f and g are real-valued functions that have limits L_1 and L_2 at c, and b is a real number, then $\lim\limits_{x\to c}(f+g) = L_1 + L_2$, $\lim\limits_{x\to c}(f \cdot g) = L_1 \cdot L_2$, $\lim\limits_{x\to c}(b \cdot f) = b \cdot L_1$, and $\lim\limits_{x\to c}(\frac{f}{g}) = \frac{L_1}{L_2}$ (provided $g(x) \neq 0$ for any x and provided $L_2 \neq 0$).

Theorem 4.4. If $f, g, h : A \to \mathbb{R}$, c is a cluster point of A, and $f(x) \leq g(x) \leq h(x)$ for all $x \neq c \in A$, then if $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} h(x) = L$, $\lim\limits_{x\to c} g(x) = L$ by the **Squeeze Theorem**.

Theorem 4.5. If f has a limit $L > 0$ at c, then there is a neighborhood of c whose image is all greater than 0; if $\lim\limits_{x\to c} f(x) = L > 0$ then there exists a $\delta > 0$ such that when $x \in B_{\mathbb{R}}(c, \delta)$, $f(x) > 0$.

Theorem 4.6. If c is a cluster point of both (c, ∞) and $(-\infty, c)$, then $\lim\limits_{x\to c} f(x) = L$ if and only if the right and left hand limits both also converge to L.

Theorem 4.7. A sequence of bounded functions is uniformly convergent if and only if for all $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that whenever $m > n > N$, it must be the case that $\|f_m - f_n\|_A = \sup\{f_m(x) - f_n(x) : x \in A\} < \varepsilon$.

4.3 Examples and Problems

9) Use epsilon-delta definition of the limit or sequential criterion for limits to establish the following:

a. $\lim_{x\to 2}\{\frac{1}{1-x}\} = -1$.

Let $\varepsilon > 0$ be given. Consider $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. Then whenever $|x-2| < \delta$, we see $|x-2| < \frac{\varepsilon}{2}$ directly from the construction of delta, and see $\left|\frac{1}{1-x}\right| < 2$ since $|x-2| < \frac{1}{2}$ implies $-\frac{1}{2} < (x-2) < \frac{1}{2}$ and so $2 > \frac{1}{x-1} > \frac{2}{3}$. So observe $\left|\frac{1}{1-x} + 1\right| = \left|\frac{x-2}{x-1}\right| = |x-2| \cdot \left|\frac{1}{x-1}\right| < \left[\frac{\varepsilon}{2}\right][2] = \varepsilon$ as required.

b. $\lim_{x\to 2}\{\frac{x}{1+x}\} = \frac{1}{2}$.

Let $\varepsilon > 0$ be given. Consider $\delta = \min\{1, 2\varepsilon\}$.Then whenever $|x-2| < \delta$, we see $|x-2| < 2\varepsilon$ and $\left|\frac{1}{2+2x}\right| < \frac{1}{2}$. The second inequality follows from $|x-1| < 1$ implying $-1 < (x-1) < 1$ and so $2 < (2x+2) < 6$ before finally $\frac{1}{2} > \frac{1}{2+2x} > \frac{1}{6}$. So observe $\left|\frac{x}{1+x} - \frac{1}{2}\right| = \left|\frac{x-1}{2+2x}\right| = |x-1| \cdot \left|\frac{1}{2x+2}\right| < [2\varepsilon]\left[\frac{1}{2}\right] = \varepsilon$ as required.

c. $\lim_{x\to 0}\{\frac{x^2}{|x|}\} = 0$.

Let $\varepsilon > 0$ be given. Consider $\delta = \varepsilon$. Then whenever $|x-0| < \delta$, clearly $\left|\frac{x^2}{|x|} - 0\right| = |x| < \varepsilon$ as required.

d. $\lim_{x\to 1}\{\frac{x^2-x+1}{x+1}\} = \frac{1}{2}$.

Let $\varepsilon > 0$ be given. Consider $\delta = \min\{1, \frac{2\varepsilon}{3}\}$. Then whenever $|x-1| < \delta$, we see $|x-1| < \frac{3\varepsilon}{2}$ and $\left|\frac{2x-1}{2x+2}\right| < \frac{3}{2}$. The second inequality follows from $|x-1| < 1$ implying $-1 < (x-1) < 1$ and so both $-1 < (2x-1) < 3$ and $2 < (2x+2) < 6$. So observe $\left|\frac{x^2-x+1}{1+x} - \frac{1}{2}\right| = \left|\frac{2x^2-3x+1}{2x+2}\right| = |x-1|\left|\frac{2x-1}{2x+2}\right| < \left[\frac{2\varepsilon}{3}\right]\left[\frac{3}{2}\right] = \varepsilon$ as required.

12) Show that the following limits do not exist:

a. $\lim_{x\to 0}\{\frac{1}{x^2}\}$.

Consider the sequence $a_n = \frac{1}{n}$. We have previously shown that $\{a_n\}_{n=1}^{\infty}$ converges to 0. But $\left\{\frac{1}{a_n^2}\right\}_{n=1}^{\infty} = \left\{\frac{1}{\left(\frac{1}{n}\right)^2}\right\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$ which is clearly divergent. So the limit doesn't exist at 0.

b. $\lim_{x\to 0}\{\frac{1}{\sqrt{x}}\}$.

The sequence $a_n = \frac{1}{n^2}$ converges to 0, but the sequence $\frac{1}{\sqrt{\frac{1}{n^2}}} = \frac{1}{1/n} = n$ clearly diverges.

c. $\lim_{x\to 0}\{x + \operatorname{sgn}(x)\}$.

The limit of the sum is the sum of the limit, so this is equivalent to showing $\lim_{x\to 0}\{x\} + \lim_{x\to 0}\{\operatorname{sgn}(x)\}$. We can show a sequence a_n that converges to 0, but where $\{\operatorname{sgn}(x)\}$ does not converge. Explicitly, consider the sequence $a_n = \frac{(-1)^n}{n}$. See $\{a_n\}_{n=1}^{\infty}$ converges to 0, but since $\operatorname{sgn}(x) = \frac{x}{|x|}$ whenever $x \neq 0$, $\operatorname{sgn}(a_n) = \frac{a_n}{|a_n|} = \frac{(-1)^n/n}{1/n} = (-1)^n$, which diverges.

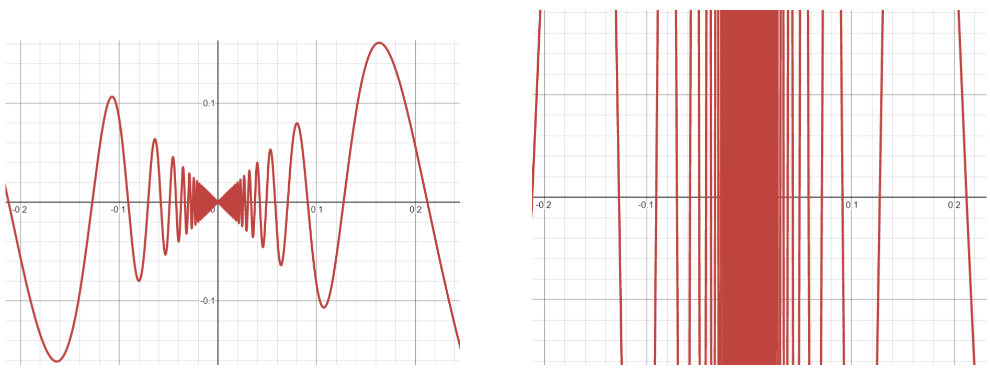
d. $\lim_{x\to 0}\{\sin\left(\frac{1}{x^2}\right)\}$.

Consider the sequence $a_n = \frac{1}{n}$. We have previously shown that $\{a_n\}_{n=1}^{\infty}$ converges to 0. But $\left\{\sin\left(\frac{1}{\left(\frac{1}{x}\right)^2}\right)\right\} = \{\sin(x^2)\}$ which is divergent (since there is no value of N whereby $\sin(m)$ and $\sin(n)$ are guaranteed to be arbitrarily close whenever $m > n > N$).

4) Prove that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist, but that $\lim_{x \to 0} x \cdot \cos\left(\frac{1}{x}\right) = 0$.

We can show the first function diverges with the sequential criteria for limits. Consider the sequence $a_n = \{\frac{1}{n\pi}\}$. Clearly this sequence converges to 0 as n tends to infinity. But see the sequence $\{f(a_n)\}$ is divergent since $\lim_{n\to\infty} \left\{ \cos\left(\frac{1}{1/n\pi}\right) \right\} = \lim_{n\to\infty} \{\cos(n\pi)\}$ (the values alternate between 1 and -1).

To show the second function converges, we use the squeeze theorem for limits. We know that for any $x \in \mathbb{R}$, $|\cos(x)| \leq 1$, so it must be the case that $-1 \leq \cos(\frac{1}{x}) \leq 1$ and therefore $-x \leq x \cdot \cos(\frac{1}{x}) \leq x$ when $x \neq 0$. Since $\lim_{x\to 0}\{-x\} = \lim_{x\to 0}\{x\} = 0$, we have $\lim_{x\to 0} x \cdot \cos\left(\frac{1}{x}\right) = 0$ as required.



(a) $x \cdot \cos(\frac{1}{x})$ (b) $\cos(\frac{1}{x})$

12) Let $f : \mathbb{R} \to \mathbb{R}$ be operation preserving. Assume $\lim_{x\to 0} f(x) = L$ exists. Prove $L = 0$ and then show that f has a limit at every point $a \in \mathbb{R}$:

Since f is operation preserving, $f(x + x) = f(x) + f(x) = 2 \cdot f(x)$. Then $\lim_{x\to 0} f(x + x) = 2 \cdot \lim_{x\to 0} f(x)$. As x tends to 0, so too does $x + x$, so we have $\lim_{x\to 0} f(x) = 2 \cdot \lim_{x\to 0} f(x)$. We know $\lim_{x\to 0} f(x) = L$, so have $L = 2L$ and therefore $L = 0$.

Let c be any point in the domain. Then $f(x - c) = f(x) - f(c)$ since f is operation preserving. So $\lim_{x\to c} f(x) = \lim_{x\to c} f(x - c) + \lim_{x\to c} f(c)$. But as x approaches c, $x - c$ tends to 0. So we have $\lim_{x\to c} f(x) = \lim_{x\to c} f(0) + \lim_{x\to c} f(c) = c$ as required.

10) Show that $f_n(x) = \cos(\pi x)^{2n}$ converges for all $x \in \mathbb{R}$.

The sequence of functions converges pointwise to $f(x) = \begin{cases} 0, & x \in \mathbb{R}\backslash\mathbb{Z} \\ 1, & x \in \mathbb{Z} \end{cases}$. Let $\varepsilon > 0$ be given and consider $N = \log_{\cos^2(\pi x)} \varepsilon$.

Then when $x \in \mathbb{Z}$, $|f_n(x) - f(x)| = |\cos(\pi x)^{2n} - 1| = |1 - 1| < \varepsilon$ and when $x \in \mathbb{R}\backslash\mathbb{Z}$ and $n > N$, $|f_n(x) - f(x)| = |\cos^2(\pi x)^n - 0| \leq \left|\cos^2(\pi x)^N\right| = \left|\cos^2(\pi x)^{\left[\log_{\cos^2(\pi x)} \varepsilon\right]}\right| = \varepsilon$

11) Show that if $a > 0$, then $f_n(x) = \frac{x}{x+n}$ converges uniformly on $[0, a]$ but not on $[0, \infty]$.

We claim the sequence of functions converges uniformly to $f(x) = 0$ on $[0, a]$. Let $\varepsilon > 0$ and $x \in [0, a]$ be given. Consider $N{=}\frac{a}{\varepsilon}$. Then whenever $n \geq N$, $|f_n(x) - f(x)| = \left|\frac{x}{x+n} - 0\right| \leq \left|\frac{a}{a+n}\right| = \left|a \cdot \frac{1}{a+n}\right| \leq \left|a \cdot \frac{1}{N}\right| = \left|a \cdot \frac{1}{\left[\frac{a}{\varepsilon}\right]}\right| = \varepsilon$ as required.

On the other hand, the sequence of functions doesn't converge uniformly on $[0, \infty]$. Since $\frac{x}{x+n}$ is bounded above by 1, if we can find a $N \in \mathbb{N}$ such that there is a $n \geq N$ whereby $\|f_n(x) - f(x)\|_{[0,\infty]} > 0$, then we will have shown the sequence is not uniformly convergent on the domain. Let $N \in \mathbb{N}$ be given. Then $f_n(n) = \frac{n}{n+n} = \frac{1}{2}$ and clearly the supremum of the difference is at least $\frac{1}{2} > 0$.

5 Continuity

5.1 Definitions

Definition 5.1. A function $f : A \to \mathbb{R}$ is **continuous at a point** c if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. A function is **continuous on a set** if it is continuous at every point in the set.

Definition 5.2. A function $f : A \to \mathbb{R}$ is **discontinuous at a point** c if there is a sequence $\{x_n\}$ that converges to c but where the sequence $\{f(x_n)\}$ doesn't converge to $f(c)$.

Definition 5.3. A function $f : A \to \mathbb{R}$ has an **absolute maximum** on A if there is a point $x_0 \in A$ such that $f(x_0) \geq f(x)$ for all $x \in A$. Analogous definitions hold for absolute minimums.

Definition 5.4. A function $f : A \to \mathbb{R}$ is **uniformly continuous** on A if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, u \in A$ satisfy $|x - u| < \delta$, then $|f(x) - f(u)| < \varepsilon$. A function that is uniformly continuous is continuous at every point in the domain, but a function that is continuous at every point in the domain is not necessarily uniformly continuous (for example $g(x) = \frac{1}{x}$ for $x \in \mathbb{R}_+$).

Definition 5.5. A function $f : A \to \mathbb{R}$ satisfies the **Lipschitz Criteria** if there exists a constant $K > 0$ such that for all $x, u \in A$, $|f(x) - f(u)| \leq K|x - u|$. Geometrically, this can be interpreted as the secant line connecting any two points of a function being bounded by a slope.

Definition 5.6. If $f : [a, b] \to \mathbb{R}$ is monotone increasing, then the **jump** of f at c is defined to be $j_f(c) = \lim\limits_{x \to c+} f - \lim\limits_{x \to c-} f$

5.2 Theorems

Theorem 5.1. The arithmetic of limits is as expected, with the condition that the quotient is continuous only if the denominator function is nowhere 0. Further the square root of positive continuous functions is continuous and the absolute value of continuous functions is also continuous.

Theorem 5.2. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions that are continuous at $c \in A$ and $f(c) \in B$ with $F(A) \subseteq B$. Then the composition $g \circ f : A \to \mathbb{R}$ is continuous at c.

Theorem 5.3. If $[a, b]$ is a closed bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded on $[a, b]$.

Theorem 5.4. If $[a, b]$ is a closed bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous, then f has an absolute maximum and minimum on $[a, b]$.

Theorem 5.5. If $[a, b]$ is a closed bounded interval and $f : [a, b] \to \mathbb{R}$ is continuous, then if $f(a) < 0 < f(b)$ or if $f(a) > 0 > f(b)$ then there exists a $c \in (a, b)$ such that $f(c) = 0$.

Theorem 5.6. Bolzano's Intermediate Value Theorem: If I is an interval and $f : I \to \mathbb{R}$ is continuous, then if $a, b \in I$ and if k is a point where $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

Theorem 5.7. If a function is Lipschitz on an interval, the function is uniformly continuous on the interval.

Theorem 5.8. If a function $f : [a, b] \to \mathbb{R}$ is monotone on an interval $[a, b]$, then for a value $c \in (a, b)$, $\lim\limits_{x \to c-} f = \sup\{f(x) : x < c\}$ and $\lim\limits_{x \to c+} f = \inf\{f(x) : x > c\}$. This function is continuous at f if and only if both of these limits equal $f(c)$.

Theorem 5.9. If a function $f : [a, b] \to \mathbb{R}$ is monotone, then the set of points at which f is discontinuous is countable.

Theorem 5.10. If a continuous function $f : [a, b] \to \mathbb{R}$ is strictly monotone, then it's inverse is continuous and strictly monotone.

5.3 Examples and Problems

3) Give an example of functions f and g that are both discontinuous at $c \in \mathbb{R}$ such that a) $f + g$ is continuous at c, b) fg is continous at c.

Consider the functions $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$ and $g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$.

Clearly these functions are individually discontinuous at 0, but their sum is continuous everywhere ($f+g$ is the constant function 1) and their product is continuous everywhere (fg is the constant function 0).

8) Let f and g be continuous real-valued functions and suppose that $f(r) = g(r)$ for all rational r. Is it true that $f(x) = g(x)$ for all real x?

Yes. Let c be an arbitrary irrational number. By the density of the rational numbers, we can construct a sequence $\{a_n\}$ of rational numbers that converges to c. By the continuity of f and g, we must have $\lim_{n\to\infty} \{f(a_n)\} = f(c)$ and $\lim_{n\to\infty} \{g(a_n)\} = g(c)$. Since every $a_i \in \{a_n\}$ is irrational, every $f(a_i) = g(a_i)$ by the assumption of the proof. So $f(c) = g(c)$. Since c is arbitrary, this proves $f = g$ at every real number.

1) Show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$ for $a > 0$.

Let $\varepsilon > 0$ be given. Consider $\delta = \varepsilon a^2$. Notice that for $x_0, x_1 \in [a, \infty)$, $x_0 x_1$ is at least a^2 (so $\frac{1}{x_0 x_1}$ is at most $\frac{1}{a^2}$). Then whenever $|x_0 - x_1| < \delta$, it must be the case that $|f(x_0) - f(x_1)| = \left|\frac{x_1 - x_0}{x_0 x_1}\right| = |x_0 - x_1| \left|\frac{1}{x_0 x_1}\right| < [\varepsilon a^2] \left[\frac{1}{a^2}\right] = \varepsilon$ as required.

8) Let f and g be increasing real-valued functions on an interval I with $f(x) \geq g(x)$ for all $x \in I$. If $y \in [f(I) \cap g(I)]$, show that $f^{-1}(y) < g^{-1}(y)$.

Label the interval $I = [a, b]$. Since $y \in [f(I) \cap g(I)]$, y is at most $g(b)$ and at least $f(a)$; $y \in [f(a), g(b)]$ and the inverse of y exists for both functions. Since $f(x) > g(x)$ for all $x \in I$, if $c \in [a, b]$ is the value such that $f(c) = y$, there exists an $\varepsilon > 0$ such that $g(c+\varepsilon) = y$. But this means $[g^{-1}(y) = c + \varepsilon] > [c = f^{-1}(y)]$.

8) Show that $f(x) = 2\ln(x) + \sqrt{x} - 2$ has a root in the interval $[1, 2]$. Use the bisection method and a calculator to find the root with error less than 10^{-2}.

The function is continuous and the interval is closed. Further, since $\left[0, \frac{\pi}{2}\right]$ is a closed bounded interval, and since $\cos(\frac{\pi}{2}) - \frac{\pi}{2} < 0 < \cos(0) - 0$, by Balzano's Intermediate Value Theorem, the given equation obtains a root on the interval.

```
interval=c(1,2)                #input interval#
maxerror=10^-2                 #input desired absolute error#
myfunction=function (x) {      #input desired equation to find roots#
  2*log(x)+sqrt(x)-2
}

a_n=vector(); a_n[1]=interval[1]
b_n=vector(); b_n[1]=interval[2]
mid=vector(); mid[1]=a_n[1]+(.5*(b_n[1]-a_n[1]))
value=vector()
error=vector()
i=1

while(suppressWarnings(min(error))>maxerror | length(error)==0) {
  value[i]=myfunction(mid[i])

  if(value[i]<0) {
    a_n[i+1]=mid[i]
    b_n[i+1]=b_n[i]
  } else {
    a_n[i+1]=a_n[i]
    b_n[i+1]=mid[i]
  }

  mid[i+1]=a_n[i+1]+(.5*(b_n[i+1]-a_n[i+1]))
  error[i]=diff(interval)*(1/2)^i
  i=i+1
}

mydf=data.frame(1:(i-1), a_n[-i], b_n[-i], mid[-i], value, error)
colnames(mydf)=c("n", "a_n", "b_n", "mid", "value", "error")
mydf
n     a_n      b_n      mid         value     error
1 1.00000 2.000000 1.500000  0.035675088 0.5000000
2 1.00000 1.500000 1.250000 -0.435678909 0.2500000
3 1.25000 1.500000 1.375000 -0.190488598 0.1250000
4 1.37500 1.500000 1.437500 -0.075231132 0.0625000
5 1.43750 1.500000 1.468750 -0.019256638 0.0312500
6 1.46875 1.500000 1.484375  0.008336910 0.0156250
7 1.46875 1.484375 1.476562 -0.005427617 0.0078125
```

Figure 5.1: Crude R Script To Estimate Roots

1) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in [a, b]$. Prove there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a, b]$.

Since f is continuous and $[a, b]$ is closed and bounded, $f([a, b])$ is bounded. As the image is a bounded subset, it has a greatest lower bound by the completeness property of the real numbers, call it $l = \inf\{f([a, b])\}$. Then by the definition of greatest lower bound, whenever $n \in \mathbb{N}$, then there must be a $x_n \in [a, b]$ whereby $l + \frac{1}{n} > f(x_n) \geq l$.

As $[a, b]$ is bounded, the sequence $\{x_n\}$ is bounded as well. Since the sequence is bounded, it has a convergent subsequence by the Bolzano-Weierstrass Theorem, call it $\lim_{k\to\infty}\{x_{n_k}\} = \alpha$. Also, since f is continuous, we have $f(\alpha) = \lim_{k\to\infty} f(\{x_{n_k}\})$.

From the previous inequality, it is clear that $l + \frac{1}{n_k} > f(x_{n_k}) \geq l$ for all $k \in \mathbb{N}$. Further $\lim_{k\to\infty}\{l + \frac{1}{n_k}\} = l = \lim_{k\to\infty}\{l\}$, so by the squeeze theorem, $f(\alpha) = \lim_{k\to\infty}\{f(x_{n_k})\} = l$. Since l is a lower bound, this proves that $f(x) \geq \alpha$ when $x \in [a, b]$

11) Show that $g : [0, 1] \to \mathbb{R}$, $g(x) = \sqrt{x}$ is not a Lipschitz function.

We argue by contraction, so assume there exists a $K > 0$ such that $g(x) \leq K \cdot x$ for all $x \in [0, 1]$. Since $\sqrt{x}$ attains a maximum of 1 at $x = 1$ in the domain, we would need to see $K \geq 1$ to have $\sqrt{x} \leq K \cdot x$ for all $x \in [0, 1]$. Consider $x_0 = \frac{1}{K^2+1}$. Clearly $x_0 \in [0, 1]$. Then if $\sqrt{x_0} \leq K \cdot x_0$, we would have $\frac{1}{K^2+1} \leq (K^2)\left(\frac{1}{K^2+1}\right)^2$ and $\frac{1}{1/(K^2+1)} \leq K^2$ and finally $(K^2 + 1) \leq K^2$, a contradiction.

This means that there is no $K > 0$ that enables $\left|\sqrt{x} - \sqrt{y}\right| \leq K \cdot |x - y|$ for all $x, y \in [0, 1]$ (since a choice of $y = 0$ would mean $\sqrt{x} \leq K \cdot x$) and we have proved that g does not meet the Lipschitz criteria.

3) Show the following are not uniformly continuous:

a. $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^2$.

To prove the function is not uniformly continuous, we go directly for the negation of the definition. So consider $\varepsilon = 1$ and let $\delta > 0$ be given. We would like to find values $p, q \in [1, \infty)$ such that $|p - q| < \delta$ but $|p^2 - q^2| \geq \varepsilon = 1$. Take $p = 1 + \frac{\frac{2}{\delta} - \frac{\delta}{2}}{2}$ and $q = p + \frac{\delta}{2}$. Clearly $|p - q| = \left|p - \left[p + \frac{\delta}{2}\right]\right| = \frac{\delta}{2} < \delta$. Then $|p^2 - q^2| = |p - q| \cdot |p + q| = \left[\frac{\delta}{2}\right] \cdot \left|2p + \frac{\delta}{2}\right| > \frac{\delta}{2} \cdot \left[\frac{2}{\delta}\right] = 1 = \varepsilon$ as required since when $p > \frac{\frac{2}{\delta} - \frac{\delta}{2}}{2}$, $2p > \frac{2}{\delta} - \frac{\delta}{2}$ and so $\left(2p + \frac{\delta}{2}\right) > \frac{2}{\delta}$.

b. $g : (0, \infty) \to \mathbb{R}$, $g(x) = \sin\left(\frac{1}{x}\right)$.

To prove g is not uniformly continuous, we can find sequences X and Y such that $X - Y = 0$ but $g(X) - g(Y) > 0$. Consider the sequences $X = \{\frac{1}{n+\pi}\}_{n \in \mathbb{N}}$ and $Y = \{\frac{1}{n}\}_{n \in \mathbb{N}}$. Clearly X and Y both tend to zero as n goes to infinity, so their difference does as well. But see $\left|\sin\left(\frac{1}{X}\right) - \sin\left(\frac{1}{Y}\right)\right| = |\sin(n + \pi) - \sin(n)| = \left|2\sin\left(\frac{(n+\pi)-n}{2}\right)\cos\left(\frac{(n+\pi)+n}{2}\right)\right| = \left|2\cos\left(n + \frac{\pi}{2}\right)\right| > 0$ for all $n \in \mathbb{N}$.

8) Prove that if f and g are uniformly continuous on $\mathbb{R}$, then $f \circ g$ is uniformly continuous.

Let $\varepsilon > 0$ be given. By the uniform continuity of f, there exists a $\delta_1 > 0$ such that whenever $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$, $|f(x) - f(y)| < \varepsilon$. By the uniform continuity of g, there exists a $\delta_2 > 0$ such that $p, q \in \mathbb{R}$ with $|p - q| < \delta_2$, $|g(p) - g(q)| < \delta_1$. Consider $\delta = \delta_2$. Then whenever $|p - q| < \delta$, $|g(p) - g(y)| < \delta_1$, and so $|f(g(p)) - f(g(q))| < \varepsilon$ as required.

6 Differentiation

6.1 Definition

Definition 6.1. A function $f : [a,b] \to \mathbb{R}$ has a **derivative** L at a point c if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x-c| < \delta$, it must be the case that $\left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon$. We denote the derivative as $f'(c) = \lim\limits_{x\to c} \frac{f(x)-f(c)}{x-c}$.

Definition 6.2. The limit of the quotient $\frac{f(x)}{g(x)}$ is said to be **indeterminate** at c if $f(c) = g(c) = 0$ or if $f(c) = g(c) = \infty$. In such cases, we can sometimes find the limit in accordance with the theorem from L'Hospitial.

Definition 6.3. The $\boldsymbol{n^{th}}$ **Taylor Polynomial For $\boldsymbol{f}$ at $\boldsymbol{x_0}$**, denoted P_n, is the polynomial $P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, whose terms provides increasingly better approximations for $f(x)$ near x_0.

Definition 6.4. A function $f : [a,b] \to \mathbb{R}$ is **convex** on the interval if for any t satisying $0 \leq t \leq 1$ and any points $x_1, x_2 \in [a,b]$, we have $f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$

6.2 Theorems

Theorem 6.1. Differentiability implies Continuity: If a function is differentiable at a point, it is continuous at that point.

Theorem 6.2. Arithmetic Of Differentiable Functions: For differentiable functions f and g, the sum $(f+g)'(c) = f'(c) + g'(c)$, the product $(fg)'(c) = f'(c)\,g(c) + f(c)\,g'(c)$, and the quotient $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$ provided $g(c) \neq 0$.

Theorem 6.3. Carathéodory's Theorem: Let f be defined on an interval containing c. Then f is differentiable at c if and only if there exists a function ϕ on I that is continuous at c satisfying $f(x) - f(c) = \phi(x)(x-c)$.

Theorem 6.4. Chain Rule: When $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ are functions such that $f(I) \subseteq J$ and f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c, and is $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Theorem 6.5. Inverse Derivative: When $f : I \to \mathbb{R}$ is a continuous strictly monotone function, and $G : f(I) \to \mathbb{R}$ is the continuous strictly monotone function inverse to f, then if f is differentiable at c and $f'(c) \neq 0$, g is differentiable at $d = f(c)$ and $g'(d) = \frac{1}{f'(g(d))}$. Similarly if f is differentiable on I with $f'(x) \neq 0$, then g is differentiable on $f(I)$ and $g'(x) = \frac{1}{f' \circ g(x)}$

Theorem 6.6. Rolle's Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous and f' exists at every point in (a,b), then if $f(a) = f(b) = k$, then there exists a point $c \in (a,b)$ such that $f(c) = k$.

Theorem 6.7. Mean Value Theorem: If $f : [a,b] \to \mathbb{R}$ is continuous and f' exists for all values in (a,b), then there is a point c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Theorem 6.8. Darboux's Theorem: If $f : [a,b] \to \mathbb{R}$ is everywhere differentiable, and if k is a number between $f'(a)$ and $f'(b)$, then there is a point $c \in (a,b)$ such that $f'(c) = k$

Theorem 6.9. Cauchy's Mean Value Theorem: When f and g are continuous functions on $[a,b]$ and differentiable on (a,b) with $g'(x)$ for all $x \in (a,b)$, then there exists a $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Theorem 6.10. L'Hospital's Rule: If f and g are differentiable functions on an interval (a, b) with $g'(x) \neq 0$ and either $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$ or $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = \pm\infty$, then if $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

Theorem 6.11. Taylor's Theorem: If f and g are differentiable functions on an interval (a, b) with $g'(x) \neq 0$ and either $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$ or $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = \pm\infty$, then if $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.

Theorem 6.12. If f has a second derivative on an open interval, then f is a convex function on the interval if and only if $f''(x) \geq 0$ for all x in the interval.

6.3 Examples and Problems

4) Find $f'(0)$ for the function $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

We claim $f'(0) = 0$. Let $\varepsilon > 0$ be given and consider $\delta = \varepsilon$. Then whenever $|x - 0| < \delta$, it must be the case that $\left|\frac{f(x)-f(0)}{x-0} - 0\right| \leq \left|\frac{x^2-0}{x}\right| = |x| < [\varepsilon]$ as required since as x approaches 0, $f(x)$ is at most x^2.

1) For each of the following real-valued functions, find the points of relative extrema, the intervals on which the function is increasing, and the intervals on which it is decreasing.

a. $f(x) = x^2 - 3x + 5$. Since the function is a differentiable polynomial on an infinite domain, we can just test values of the derivative. See $f'(x) = 2x - 3$, so f is increasing on $x > \frac{3}{2}$ (since $2x - 3 > 0 \implies x > \frac{3}{2}$), decreasing on $x < \frac{3}{2}$ (since $2x - 3 < 0 \implies x < \frac{3}{2}$), and has extrema at $x = \frac{3}{2}$ by the first derivative test.

4) Show that if $x > 0$, then $1 + \frac{1}{x} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{1}{x}$.

Taylor's Theorem tells us that for a $k+1$ times differentiable real-valued function f, there exists a $c \in (x, x_0)$ such that $f(x) = \left(\sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n\right) + \left(\frac{f^{(n+1)}(c)}{n+1!} \cdot (x - x_0)^{n+1}\right)$. It is clear that $\sqrt{1+x}$ is at least three-times differentiable. So we have $f(x) = \sqrt{1+x} = \left(\frac{f(x_0)}{0!} \cdot (x - x_0)^0\right) + \left(\frac{f'(x_0)}{1!} \cdot (x - x_0)^1\right) + \left(\frac{f''(x_0)}{2!} \cdot (x - x_0)^2\right)$ Centering the function at zero (the MacLaurin Series), we have $f(x) = \left(\sqrt{1}\right) + \left(\frac{1}{2} \cdot x\right) + \left(\frac{1/4}{2!} \cdot x^2\right) = 1 + \frac{1}{x} + \frac{x^2}{8}$. Subtracting the third term gives the expected result.

9) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x^4 + x^4 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that f has an absolute minimum at $x = 0$, but that its derivative has both positive and negative values in every neighborhood of 0.

First observe that $\sin\left(\frac{1}{x}\right)$ is at least -1, so at minimum, $f(x) = (2-1)x^4 = x^4$. But $x^4 > 0$ for any $x \neq 0$, so it is clear an absolute minimum is achieved at $x = 0$.

When $x \neq 0$, $f'(x) = 8x^3 + \left[4x^3 \sin\left(\frac{1}{x}\right) + x^4 \cos\left(\frac{1}{x}\right) \frac{-1}{x^2}\right] = 8x^3 + 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right)$ by the product rule and the chain rule.

Notice we can make the sine term 0 and the cosine term 1 when they have inputs in the form $2m\pi$ for some natural number m. Similarly, we can make the sine term 1 and the cosine term 0 when they have inputs in the form $(4n+1)\frac{\pi}{2}$ for some natural number n. Let r be an arbitrary positive radius. Then by the Archimedean property of the real numbers, we can find large enough $n, m \in \mathbb{N}$ such that $\frac{1}{(4n+1)\frac{\pi}{2}} < \frac{1}{2m\pi} < r$. Then $f'(\frac{1}{(4n+1)\frac{\pi}{2}})$ is positive since the third term disappears and the first two terms are positive. On the other hand, $f'\left(\frac{1}{2m\pi}\right)$ is negative since the second term disappears, and since the third term dominates the first for sufficiently small x. Since r was arbitrary, this proves that the derivative in any neighborhood of 0 has both positive and negative values.

13) Let $f : [a, b] \to \mathbb{R}$ be differentiable. If f' is positive on the interval, show that f is strictly monotone increasing on the interval.

Let x_1, x_2 be two points in $[a, b]$ with $x_1 < x_2$. By the mean value theorem, there is a point $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $f'(x)$ is positive for all $x \in [a, b]$, $f'(c)$ is positive. Since $x_2 > x_1$, this means $f(x_2) > f(x_1)$. But since x_2 and x_1 were arbitrary, this is precisely the definition of monotone increasing.

10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Show that $g'(0) = 1$, but that g is not monotonic in any neighborhood of 0.

We see g is an enveloping function:

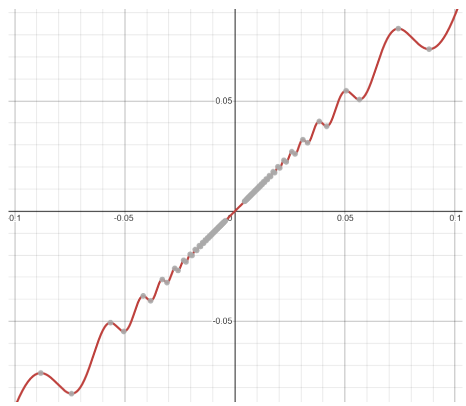


Figure 6.1: $g(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

We have $g'(0) = \lim\limits_{x\to 0} \frac{g(x)-g(0)}{x} = \lim\limits_{x\to 0} \frac{x+2x^2\sin(\frac{1}{x})}{x} = \lim\limits_{x\to 0} 1 + 2x\sin(\frac{1}{x}) = 1$. But when $x \neq 0$, $g'(x) = 1 + \left[4x\sin(\frac{1}{x}) + 2x^2 \cdot \left(\cos(\frac{1}{x}) \cdot \frac{-1}{x^2}\right)\right]$ by the product rule and the chain rule, which can be rewritten as $1 + 4x\sin(\frac{1}{x}) - 2\cos(\frac{1}{x})$.

Now consider the open ball centered at 0 with radius r, $B_{\mathbb{R}}(0, r)$. By the Archimedean property of the real numbers, for any arbitrary $r > 0$, one can select a large enough $n, m \in \mathbb{N}$ such that $\frac{1}{(4n+1)\frac{\pi}{2}} < \frac{1}{2m\pi} < r$. Then $g'\left(\frac{1}{(4n+1)\frac{\pi}{2}}\right) = 1 + 4\left(\frac{1}{(4n+1)\frac{\pi}{2}}\right)\sin\left((4n+1)\frac{\pi}{2}\right) - 2\cos\left((4n+1)\frac{\pi}{2}\right) = 1 + 4(\frac{1}{(4n+1)\frac{\pi}{2}}) > 1$ but $g'(\frac{1}{2m\pi}) = 1 + 4(\frac{1}{2m\pi})\sin(2m\pi) - 2\cos(2m\pi) = 1 - 2 = -1$. Since r is arbitrary, this proves any neighborhood of 0 cannot be monotonic.

7) Evaluate the following limits:

a. $\lim\limits_{x\to 0+} \frac{\ln(x+1)}{\sin(x)}$.

This is a Type I problem $(\frac{0}{0})$ associated with L'Hospital's Rule. Taking the derivative of the numerator and denominator separately, $\lim\limits_{x\to 0+} \frac{\ln(x+1)}{\sin(x)} = \lim\limits_{x\to 0+} \frac{\frac{1}{x+1}}{\cos(x)} = \frac{1}{1} = 1$

b. $\lim\limits_{x\to 0+} \frac{\ln(\cos(x))}{x}$.

This is a Type III problem $(\frac{\infty}{0})$ associated with L'Hospital's Rule. Using the Chain Rule to take the derivative in the numerator, we have $\lim\limits_{x\to 0+} \frac{\ln(\cos(x))}{x} = \lim\limits_{x\to 0+} \frac{\frac{1}{\cos(x)} \cdot -\sin(x)}{1} = -\tan(x) = 0$

c. $\lim\limits_{x\to 0+} \frac{\tan(x)-x}{x^3}$.

We can repetitively use L'Hospital's Rule, trig identities, the chain rule, and the product rule. We have $\lim\limits_{x\to 0+} \frac{\tan(x)-x}{x^3} = \lim\limits_{x\to 0+} \frac{\sec^2(x)-1}{3x^2} = \lim\limits_{x\to 0+} \frac{2\sec^2(x)\tan(x)}{6x} = \lim\limits_{x\to 0+} \frac{\left[4\sec^2(x)\tan(x)\right]\tan(x)+2\sec^2(x)\left[\sec^2(x)\right]}{6} = \frac{2}{6} = \frac{1}{3}$

d. $\lim\limits_{x\to\infty} \left(1+\frac{3}{x}\right)^x$. This is a form of Euler's number. One way to think about Euler's Number is that it gives the growth rate of a continuously compounding process of doubling (100%) over unit length. Here, the growth rate is not 100%, but 300%, so the limit is e^3.

7 Integration

7.1 Definition

Definition 7.1. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann Integrable** if there exists a L such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever the largest partition of the closed interval domain $\|P\| = \max\{(x_1 - a), (x_2 - x_1), \ldots, (b - x_n)\} < \delta$, it must be the case that the **Riemann Sum** of the partition, $S(f, P) = \sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ (where each t_i is an arbitrary point of the given sub-interval) is within ε of L; $|S(f, P) - L| < \varepsilon$. Usually, we write $L = \int_a^b f(x)dx$

Definition 7.2. An **antiderivative** of f is a function F such that $F'(x) = f(x)$ for all $x \in [a, b]$

7.2 Theorems

Theorem 7.1. If g is Riemann integrable on $[a, b]$ and if $f(x) = g(X)$ for all but a finite number of points in $[a, b]$, then f is Riemann integrable and $\int_a^b f = \int_a^b g$.

Theorem 7.2. If f is Riemann integrable on $[a, b]$, then f is bounded on $[a, b]$.

Theorem 7.3. A function $f : [a, b] \to \mathbb{R}$ is Riemann Integrable if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a, b]$ with $\|\dot{P}\| < \delta$ and $\|\dot{Q}\| < \delta$, then $\left|S(f, \dot{P}) - S(f, \dot{Q})\right| < \varepsilon$

Theorem 7.4. A function $f : [a, b] \to \mathbb{R}$ is Riemann Integrable if and only if for every $\varepsilon > 0$, there are Riemann Integrable functions α and ω with $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in [a, b]$ and such that $\int_a^b (\omega - \alpha) < \varepsilon$

Theorem 7.5. If a function is continuous on $[a, b]$, it is Riemann Integrable on $[a, b]$. If a function is monotone on $[a, b]$, it is Riemann Integrable on $[a, b]$.

Theorem 7.6. A function is Riemann Integrable on $[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are Riemann Integrable; $\int_a^b f = \int_a^c f + \int_c^b f$

Theorem 7.7. Fundamental Theorem Of Calculus: When E is a finite set in an interval $[a, b] \in \mathbb{R}$, and when $f, F : [a, b] \to \mathbb{R}$, then if F is continuous on $[a, b]$, if $F'(x) = f(x)$ for all $x \in [a, b] / E$, and if f is Riemann Integrable on $[a, b]$, then $\int_a^b f = F(b) - F(a)$

Similarly, if f is a Riemann Integrable function on $[a, b]$ that is continuous at $c \in [a, b]$, and $F(x) = \int_a^x f$ for all $x \in [a, b]$, then $F'(c) = f(c)$.

7.3 Examples and Problems

1) Prove the Fundamental Theorem of Calculus.

Assume that $f : [a, b] \to \mathbb{R}$ is a Riemann Integrable function, that $F : [a, b] \to \mathbb{R}$ is a continuous function, that E is a finite set of points from $[a, b]$, and that $F'(x) = f(x)$ for all $x \in [a, b] / E$. We would like to show that $\int_a^b f = F(b) - F(a)$.

Since E is finite, label $E = \{e_0 = a, e_1, e_2, \ldots, e_n = b\}$ and divide $[a, b]$ into n sub-intervals $I_1 = [e_0, e_1]$, $I_2 = [e_1, e_2]$, $\ldots$, $I_n = [e_{n-1}, e_n]$. Then let $\varepsilon > 0$ be given. By the definition of Riemann Integral, for each sub-interval I_j, there exists a δ_j such that for any tagged partition $\dot{P}_j$ of I_j with $\|\dot{P}_j\| < \delta_j$, it must be the case that $\left| S\left(f, \dot{P}_j\right) - \int_{e_{j-1}}^{e_j} f \right| < \varepsilon$.

Consider an arbitrary interval $I_k = [e_{k-1}, e_k]$ and label it's partition $P_k = \{[p_{i-1}, p_i]\}_{i=1}^{m}$. By supposition of the proof, F is differentiable on (e_{k-1}, e_k) and continuous on $[e_{k-1}, e_k]$. Then by the mean value theorem, in each subinterval of the partition P_j, there exists a u_{j_i} such that $\frac{F(p_i)-F(p_{i-1})}{(p_i-p_{i-1})}=F'(u_{j_i})$ where $F'(u_{j_i}) = f(u_{j_i})$ again by supposition of the proof. Choose each u_{j_i} to be the tag point of each respective sub-interval. Then $\sum_{i=1}^{m} f(u_{j_i})(p_i - p_{i-1}) = F(e_{k-1}) - F(e_k)$ by the telescoping properties of the sum. So we have $\left|(F(e_{k-1}) - F(e_k)) - \int_{e_{j-1}}^{e_j} f\right| < \varepsilon$, and since ε is arbitrarily small, that $(F(e_{k-1}) - F(e_k)) = \int_{e_{j-1}}^{e_j} f$.

By choosing $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, we see $\int_a^b f = \int_a^{e_1} f + \int_{e_1}^{e_2} f + \ldots + \int_{e_{n-1}}^{b} f = [F(e_1) - F(a)] + [F(e_2) - F(e_1)] + \cdots + [F(b) - F(e_{n-1})] = F(b) - F(a)$ and we have proved the desired result.

6) Let $f(x) = \begin{cases} 2, & x \in [0,1) \\ 1, & x \in [1,2] \end{cases}$. Show that f is Riemann Integrable on $[0,2]$ and find it's integral.

We claim $\int_0^2 f(x)dx = 3$. Let $\varepsilon > 0$ be given, consider $\delta = \frac{\varepsilon}{2}$, and let P be an arbitrary partition of $[0,2]$ with a sub-interval of at most δ.

Call P_1 the subset of P which contains tags in $[0,1)$, P_2 the subset of P which contains tags in $[1,2]$, U_1 the union of all sub-intervals of P_1, and U_2 the union of all sub-intervals of P_2. Since the width of each subinterval is at most $\delta = \frac{\varepsilon}{2}$, it is clear that U_1 is contained in the interval $[0, 1+\delta]$ and contains the interval $[0, 1-\delta]$; $[0, 1-\delta] \subset U_1 \subset [0, 1+\delta]$. Similarly we have $[1+\delta, 2] \subset U_2 \subset [1-\delta, 2]$.

This implies the Riemann Sum of f over P_1 is at least $f(x) \cdot (1-\delta)$ and at most $f(x) \cdot (1+\delta)$; $2(1-\delta) \leq S\left(f, \dot{P_1}\right) \leq 2(1+\delta)$. For the same reason, we have $1(1-\delta) \leq S\left(f, \dot{P_2}\right) \leq 1(1+\delta)$.

Clearly, $S(f,P) = S\left(f, \dot{P_1}\right) + S\left(f, \dot{P_2}\right)$. Then from the above inequalities, we can deduce $3-2\delta \leq S(f,P) \leq 3+2\delta$; $|S(f,P) - 3| \leq 2\delta = 2\left[\frac{\varepsilon}{2}\right] = \varepsilon$. This proves the claim.

12) Show that $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \in (0,1]$ and $g(0) = 0$ is Riemann Integrable on $[0,1]$.

Let $\varepsilon > 0$ be given and consider $\delta = \frac{\varepsilon}{2}$.

Consider the functions $\omega(x) = \begin{cases} g(x), & (\delta, 1] \\ 1, & [0, \delta] \end{cases}$ and $\alpha(x) = \begin{cases} g(x), & (\delta, 1] \\ -1, & [0, \delta] \end{cases}$. Since the sine function is bounded by 0 and 1, we have $\alpha(x) \leq g(x) \leq \omega(x)$. Then $\int_0^1 (\omega(x) - \alpha(x))\, dx = \int_0^\delta (\omega(x) - \alpha(x))\, dx + \int_\delta^1 (\omega(x) - \alpha(x))\, dx =$ $[\delta - (-\delta)] + [0] = 2\delta = 2\left[\frac{\varepsilon}{2}\right] = \varepsilon$. This proves g is Riemann Integrable by the squeeze theorem for integration.

5) Prove the following:

a. If $\Phi : [a, b] \to \mathbb{R}$ is an antiderivative of $f : [a, b] \to \mathbb{R}$, show that $\Phi_c(x) = \Phi(x) + c$ is also an antiderivative of f.

We'd like to show that for all $c \in \mathbb{R}$ and $x \in [a, b]$, $\Phi'_c(x) = f(x)$. Let x and c be given. See $\Phi'_c = \Phi'(x) + c'$ (the derivative of the sum is the sum of the derivative), and since c is a constant, $\Phi'_c = \Phi'(x)$. But $\Phi'(x)$ is an antiderivative of f by assumption, so $\Phi'_c = \Phi'(x) = f(x)$.

b. If Φ_1 and Φ_2 are two antiderivatives of f, prove $\Phi_1 - \Phi_2$ is constant.

Let $x_0, x_1 \in [a, b]$ be given. Since Φ_1 and Φ_2 are both differentiable functions, their difference is as well. By the mean value theorem, there exists a $c \in (x_0, x_1)$ such that $(\Phi_1 - \Phi_2)'(c) = \frac{(\Phi_1 - \Phi_2)(x_2) - (\Phi_1 - \Phi_2)(x_1)}{x_2 - x_1}$. By the definition of antiderivative, $(\Phi_1 - \Phi_2)'(c) = f(c) - f(c) = 0$. So $(\Phi_1 - \Phi_2)(x_2) - (\Phi_1 - \Phi_2)(x_1) = 0$ and we have proved the difference is a constant since x_0 and x_1 were arbitrary.