Table of Contents and Results

1. Metric Spaces

A metric space is some set E, along with some function $d(E \times E) \to \mathbb{R}$ such that the following four principles are observed:

1) $\forall p, q \in E,\ d(p,q) \geq 0$
2) $[d(p,q) = 0] \Leftrightarrow [p = q]$
2) 3) $\forall p, q \epsilon E, d(p,q) = d(q,p)$
4) $\forall p, q, r \epsilon E, d(p,r) \leq d(p,q) + d(q,r)$

The fourth condition is known as the triangle inequality. Note that the codomain of the function is the set of real numbers because it's the unique complete, totally ordered field. The background and implications of this will be left out of the paper.

The basic intuition behind a metric space is that one is able to specify an abstract "distance" between two points within the space. The set of real numbers and the function $d(p,q) = |p - q|$ is the most common metric used in 1-space, while the Euclidean Metric is the common metric used on ordered n-tuples—it is the set $\mathbb{R}^n$ along with the function $d\left(\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}\right) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$.

While the above two examples are likely the most prevalent, there are certainly other valid metrics. Take for example the Taxi-Cab Metric, which essentially measures the "distance" between two points in a plane when only working vertically and horizontally. In comparison to the Euclidean Metric, this can be thought of as a distance rather than a displacement. More formally, we have $d\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right) = |p_1 - q_1| + |p_2 - q_2|$, although this could be expanded to higher dimensions as well. Similarly, the Sup Metric measures the largest deviances between elements of vectors, $d\left(\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}\right) = max\{|p_1 - q_1|, \dots, |p_n - q_n|\}$.

For examples sake, consider two points $p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, q = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Depending on the metric used, we see wildly different distance: the Euclidean Metric yields $\sqrt{14}$, the Taxi-Cab Metric 6, the Sup Metric 3, and the discrete metric (where $d(p,q) = 1$ whenever $p \neq q$) 1.

2. Open, Closed, and Connected Sets

Let (E, d) be a generic metric space where $p_0 \in E$ and $r \in \mathbb{R}_+$. Then an open ball centered at p_0 with radius r is denoted $B_E(p_0, r)$ and is defined as $\{p \in E: d(p, p_0) < r\}$. It is important to observe that the condition of the set is strictly less than (endpoints are not included). If instead the condition was less than or equal to, we would have a closed ball denoted $\overline{B_E}(p_0, r)$. The open ball is the generally preferred abstraction of an open interval (on a line) or an open disk (on a plane). The definition of an open ball is used in the definition of open sets.

Let S be a subset of a metric space (E, d). Then S is an open set if $\forall s \in S, \exists r \in \mathbb{R}_+$ such that $B_E(s, r) \subseteq S$. To see why this definition makes sense, we contrast an elementary non-example with an elementary example in one-space.

First the non-example. Take the set $S_1 = \{r: r \in [0,1]\}$. This is of course a subset of the real numbers, which forms a metric space under the absolute value function. To show this set isn't open, we reverse the qualifiers and negate the conclusion. That is, we must show that $\exists s \in S_1$ such that $\forall r \in \mathbb{R}_+, B_E(s, r) \nsubseteq S_1$. It is clear that regardless of the choice of r, the open ball centered at an end point will contain values outside of S_1. Meanwhile, the set $S_2 = \{r: r \in (0,1)\}$ is open. Select any $s \in S_2$ and observe that if $r = \frac{min\{|1-s|,|0,s|\}}{2}$, then the open ball centered at s with radius r will always be within the open interval. For concreteness, we'll show a specific example, say .999. Then if $r = \frac{|1-.999|}{2} = .0005$, the ball centered around .999 will be the open interval $(0.9985, 0.9995) \subseteq (0,1)$.

Certain properties of open sets are enjoyed by all metric spaces. We will state without proof the following properties:

1) The null set is open
2) The full set is open
3) The union of an arbitrary number of open sets is open
4) The intersection of a finite number of open sets is open

In addition, see that all open balls are open sets. It is important to note that a subset of a metric space is bounded if it is a subset of an open ball centered at a point in the set.

A set S is connected if whenever S is a subset of the union of disjoint, open, and non-empty sets U and V, either the intersection of S and U is the null set, or the intersection of S and V is the null set. Therefore to call S disconnected means that there exists disjoint, open, and non-empty sets U and V where $S \subseteq U \cup V$ and both $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$.

For example, the set of rational numbers $\mathbb{Q} = \left\{\frac{m}{n}: (m, n \in \mathbb{Z})^\wedge(n \neq 0)\right\}$ is disconnected inside the real numbers. Consider the open and disjoint sets $\left(-\infty, \sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$. The union of the sets covers the rational numbers since $\sqrt{2}$ is not rational. To see this is the case, we argue by contradiction, and assume that $\sqrt{2} = \frac{m}{n}$ for some coprime integers m and n. Then $2 =$

$\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$, and $2n^2 = m^2$. So m^2 is even since it is double the value of a natural number. The square of odd numbers is odd, so m itself must be even, and can be written in the form $m = 2k$ for some integer k. We then have $2n^2 = (2k)^2 = 4k^2$, and so can say $n^2 = 2k^2$. By the same logic as before, we know n is even. But if m and n are both even, they cannot be coprime. Of course, the intersection of the rational numbers and $(-\infty, \sqrt{2})$ is not the null set (consider the value 1). Similarly the intersection of the rational numbers and $(\sqrt{2}, \infty)$ is not the null set (consider the value 2) and we've reached our conclusion.

For a positive example, the closed real interval $[a, b]$ is connected inside the real numbers. To see this is the case, see Theorem 2.1 below.

Theorem 2.1: S is connected in $\mathbb{R}$ if and only if for any distinct a, b in S with $a < b$, $C = \{c \in \mathbb{R} | a < c < b,\ c \in \mathbb{R}\} \subseteq S$.

Pf: First assume that S is connected. We'd like to show that S has the above property (henceforth "betweeness"). If S didn't have the betweeness property, then there would be some $\hat{c} \in \mathrm{C}$ that isn't in S for a given a and b. Then let $U = (-\infty,\ \hat{c})$ and $V = (\hat{c}, \infty)$. Clearly U and V cover S and are open and disjoint. Since $a < \hat{c} < b$, a must be in U and b must be in V. But since a and b are members of S, $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$, which means S would be disconnected and we've reached our contradiction.

Now assume S has the betweeness property. We'd like to show S is connected. Let U and V be disjoint, open, and non-empty sets that cover S. Then proceed by contradiction and imagine that $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$ so S is disconnected. Now let a be a member of U and b a member of V with $a < b$ and consider $[a, b] \cap \bar{V}$, where $\bar{V}$ is defined to be the compliment of V. Both sets are closed, so the intersection of the sets is closed as well. Since S has the betweeness property by assumption, $[a, b] \subseteq S$. So, since U and V cover S, they certainly cover $[a, b]$ which means the intersection of $[a, b]$ and $\bar{V}$ is a subset of U. Now see that b is the least upper bound for $[a, b] \cap \bar{V}$. But $[a, b] \cap \bar{V} \subseteq U$, which is open. By the definition of openness, $\exists r \in \mathbb{R}_+$ such that $B_{\mathbb{R}}(b, r) \subseteq U$, so b can't be a least upper bound.

3. Limits and Continuity

One of the first topics that's studied by a student in elementary calculus is the idea of continuity. Formally, a function between two metric spaces $f: E \to \hat{E}$ is continuous at some point in the domain p_0 if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any point $p \in E$ within the open ball centered at p_0 with radius δ, we know that $f(p)$ is an element of the open ball centered at $f(p_0)$ with radius ε. What this is essentially saying is that one can force outputs of the function to be within a specified distance around a point by specifying how close inputs need to be to a point. Possibly a better way to write this, one that would be more conducive to proof writing, is $d(p_0, p) < \delta \Rightarrow \hat{d}\big(f(p_0), f(p)\big) < \varepsilon$. While these definitions use the idea of an open ball, there is another way to show that a function in continuous that uses the idea of openness in a broader sense.

Theorem 3.1 A function $f: E_1 \to E_2$ is continuous if and only if the inverse image of any open set in the codomain is an open set in the domain.

Pf: First assume the function is continuous and let V be an open set in the codomain (we know such a set exists since the full set is itself open). We want to show that $U = f^{-1}(V)$ is open as well. To do so, we aim for the definition: that for any arbitrary $p_0 \in U$ there is some radius value r so that $B_{E_1}(p_0, r) \subseteq U$. Let $p_0 \in U$ be given. This means $f(p_0) \in f(U) = V$. By the definition of the openness of V, there is some value $r^* > 0$ where $B_{E_2}(f(p_0), r^*) \subseteq V$. By the continuity of f at p_0, we know that for any $\varepsilon > 0$ (and in particular $\varepsilon = r^*$) there is some value $\delta > 0$ that forces $f(p) \in B_{E_2}(f(p_0), r^*) \subseteq V$ whenever $p \in B_{E_1}(p_0, \delta)$. Stated differently, $f\left(B_{E_1}(p_0, \delta)\right) \subseteq B_{E_2}(f(p_0), r^*) \subseteq V$. Applying the inverse function, we have $B_{E_1}(p_0, \delta) \subseteq f^{-1}(V) = U$, so by choosing $r = \delta$, we have shown U is open.

Now assume that the inverse image of any open set in the codomain is open in the domain. We want to show the function is continuous. Let $\varepsilon > 0$ be given and let p_0 be any arbitrary point in E_1. Then $B_{E_2}(f(p_0), \varepsilon)$ is open (open balls are open sets). Call the ball V. By assumption, $f^{-1}(V)$ is open in E_1. By the definition of openness, this means that there is a radius value r^{**} such that $B_{E_1}(p_0, r^{**}) \subseteq f^{-1}(V) = f^{-1}\left(B_{E_2}(f(p_0), \varepsilon)\right)$. Then $\forall p \in B_{E_1}(p_0, r^{**}) \subseteq f^{-1}(V)$, we have $f(p) \in V = B_{E_2}(f(p_0), \varepsilon)$ as required and have shown the continuity of the function.

We can also determine if a function is continuous via the limit definition. First, we state the definition of the limit of a function. Where $f: E_1 \to E_2$ is a map between metric spaces, $\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni \ 0 < d_1(x - x_0) < \delta \Rightarrow d_2(f(x), L) < \varepsilon$. L is called the limit of f as x approaches x_0, and it is a unique value. It is natural to assume that the distance metric is absolute value for single-variable real functions. Notice that directly following this definition, we can claim that a function f is continuous at a punctured interval around a if $\lim_{x \to a} f(x) = f(a)$.

Next, we show some properties of limits and continuous functions:

Lemma 3.1 the limit of a constant multiplied by a function is equal to the constant multiplied by the limit of the function.

Pf: Let $\lim_{x\to a} f(x) = L$, and let C be some constant term. We would like to show that $\lim_{x\to a} f(x) = (C \cdot f(x)) = C \cdot \lim_{x\to a} f(x) = C \cdot L$. Since $\lim_{x\to a} f(x) = L$, we know there is some $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it must be the case that $|f(x) - L| < \varepsilon$ for any positive value of ε. Let δ_1 be the value which forces $|f(x) - L| < \frac{\varepsilon}{|C|}$. We want to show the existence of a δ_2 value that forces $|Cf(x) - CL| < \varepsilon$, regardless of how small epsilon is, whenever $0 < |x - a| < \delta_2$. Consider $\delta_2 = \delta_1$. Then when $0 < |x - a| < \delta_2 = \delta_1$, $|Cf(x) - CL| = |C| \cdot |f(x) - L| < |C| \cdot \frac{\varepsilon}{|C|} = \varepsilon$ and we are done. Note that the last inequality is a result of the construction of δ_1.

Lemma 3.2 the limit of the sum is equal to the sum of the limits.

Pf: Let $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. We'd like to show $\lim_{x\to a}[\, f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$. So let $\varepsilon > 0$ be given. We want to specify a value $\delta > 0$ that will ensure $|(f(x) + g(x)) - (L_1 + L_2)| < \varepsilon$ whenever $0 < |x - a| < \delta$. First observe that there is some $\delta_1 > 0$ that forces $|f(x) - L_1| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_1$ by the definition of a limit. Similarly, there is some $\delta_2 > 0$ that forces $|g(x) - L_2| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$. So select $\delta = \max\{\delta_1, \delta_2\}$.

We know $|(f(x) + g(x)) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)|$, which by the property of absolute value is less than or equal to $|f(x) - L_1| + |g(x) - L_2|$. By the construction of δ, $|f(x) - L_1| < \frac{\varepsilon}{2}$ and $|g(x) - L_2| < \frac{\varepsilon}{2}$, so $|(f(x) + g(x)) - (L_1 + L_2)| < \varepsilon$ as required.

Lemma 3.3 $\lim_{x\to a}\big[f\big(g(x)\big)\big] = f\left(\lim_{x\to a}[g(x)]\right)$, assuming both $\lim_{x\to a}[g(x)]$ exists and f is continuous at $\lim_{x\to a}[g(x)]$.

Pf: For notational ease, call $\lim_{x\to a}[g(x)] = B$. Let ε be given. Going directly for the definition, we need to find a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $\big|f\big(g(x)\big) - f(B)\big| < \varepsilon$. By the continuity of f at B, we know. $\lim_{g(x)\to B}\big[f\big(g(x)\big)\big] = f(B)$ So there exists some positive value δ_1 wherein $\big|\big[f\big(g(x)\big)\big] - f(B)\big| < \varepsilon$ whenever $0 < |g(x) - B| < \delta_1$. Since $\lim_{x\to a}[g(x)] = B$, we know that for any positive value, namely δ_1, we can find some value $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|g(x) - B| < \delta_1$.

Then recognize with a choice of $\delta = \delta_2$, $0 < |x - a| < \delta$ implies $|g(x) - B| < \delta_1$, which in turn implies $|[f(g(x))] - f(B)| < \varepsilon$ as required.

Lemma 3.4 Polynomial functions are continuous.

Pf: We would like to show that any function in the form $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$, where each c_i is a real number and n a natural number, is continuous. To do so, we will prove a couple of additional lemmas in succession to arrive at our result. Let $\varepsilon > 0$ be given.

Lemma 3.4.1 The constant function is continuous.

Pf: So for some function $f_1(x) = c_0$, we'd like to prove that $\lim_{x \to a} f_1(x) = f_1(a)$. We need to specify a $\delta > 0$ so that $0 < |x - a| < \delta$ implies $|f_1(x) - f_1(a)| < \varepsilon$. Choose $\delta = \varepsilon$. Since $\varepsilon > 0$, $|f_1(x) - f_1(a)| = |c_0 - c_0| = 0 < \varepsilon$ as required.

Lemma 3.4.2 The trivial function $f_2(x) = x$ is continuous.
Pf: Once again consider $\delta = \varepsilon$. Such a choice means that whenever $0 < |x - a| < \delta, |f_2(x) - f_2(a)| = |x - a| < \delta = \varepsilon$ and we are done.

Lemma 3.4.3 The sum of continuous functions are continuous.

Pf: Let $\lim_{x \to a} f_3(x) = f_3(a)$ and $\lim_{x \to a} f_4(x) = f_4(a)$. We want to show $\lim_{x \to a}[f_3(x) + f_4(x)] = f_3(a) + f_4(a)$, which we can accomplish by specifying a value $\delta > 0$ that will ensure $|(f_3(x) + f_4(x)) - (f_3(a) + f_4(a))| < \varepsilon$ whenever $0 < |x - a| < \delta$. First observe that there is some $\delta_1 > 0$ that forces $|f_3(x) - f_3(a)| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_1$ by the definition of a limit. Similarly, there is some $\delta_2 > 0$ that forces $|f_4(x) - f_4(a)| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$. So select $\delta = \max\{\delta_1, \delta_2\}$.We know $|(f_3(x) + f_4(x)) - (f_3(a) + f_4(a))| = |(f_3(x) - f_3(a)) + (f_4(x) - f_4(a))|$, which by the property of absolute value is less than or equal to $|f_3(x) - f_3(a)| + |f_4(x) - f_4(a)|$. By the construction of δ, $|f_3(x) - f_3(a)| < \frac{\varepsilon}{2}$ and $|f_4(x) - f_4(a)| < \frac{\varepsilon}{2}$, so $|(f_3(x) + f_4(x)) - (f_3(a) + f_4(a))| < \varepsilon$ as required. Applying this to Lemma 1.1 and Lemma 1.2, this means that any polynomial in the form $p(x) = x + c_0$ is continuous.

Lemma 3.4.4 The product of continuous functions are continuous.

Pf: Consider the two functions in Lemma 3.4.3. We need to show $\lim_{x \to a}[f_3(x) \cdot f_4(x)] = f_3(a) \cdot f_4(a)$. Our goal is to specify a value $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it must be the case that $|(f_3(x) \cdot f_4(x)) - (f_3(a) \cdot f_4(a))| < \varepsilon$. By the continuity of f_3, we know that for an arbitrary positive value, namely $\frac{\varepsilon}{2 \cdot (\varepsilon + |f_4(a)|)}$, there is some value $\delta_3 > 0$ whereby $|f_3(x) - f_3(a)| < \frac{\varepsilon}{2 \cdot (\varepsilon + |f_4(a)|)}$ whenever $0 < |x - a| < \delta_3$. Note that $f_4(a)$ is just a constant. By the continuity of f_4, we know that there is a $\delta_4 > 0$ such that whenever $0 < |x - a| < \delta_4$ we have $|f_4(x) - f_4(a)| < \varepsilon$. By the triangle inequality, this means that $|f_4(x) - f_4(a)| \leq |f_4(x)| - |f_4(a)| < \varepsilon$, so $|f_4(x)| < \varepsilon + |f_4(a)|$. Further, we know there is some $\delta_5 > 0$ that forces $|f_4(x) - f_4(a)| < \frac{\varepsilon}{2\cdot|f_3(a)|+1}$ (to avoid the case where $f_3(a) = 0$, we add the 1) whenever $0 < |x - a| < \delta_5$. By selecting $\delta = min\{\delta_3, \delta_4, \delta_5\}$, we will ensure that each of the above three conclusions is realized: $|f_3(x) - f_3(a)| < \frac{\varepsilon}{2\cdot(\varepsilon+|f_4(a)|)}$, $|f_4(x)| < \varepsilon + |f_4(a)|$, and $|f_4(x) - f_4(a)| < \frac{\varepsilon}{2\cdot|f_3(a)|+1}$. Now assume that $0 < |x - a| < \delta$, and observe

$\left|\left(f_3(x) \cdot f_4(x)\right) - \left(f_3(a) \cdot f_4(a)\right)\right| =$

$\left|\left(f_3(x)f_4(x)\right) - \left(f_4(x)f_3(a) - f_4(x)f_3(a)\right) - \left(f_3(a)f_4(a)\right)\right|$ by adding zero

$= \left|\left(f_3(x)f_4(x) - f_4(x)f_3(a)\right) + \left(f_4(x)f_3(a) - f_3(a)f_4(a)\right)\right|$ by grouping terms

$= \left|\left(\left(f_4(x)\right)\left(f_3(x) - f_3(a)\right)\right) + \left(\left(f_3(a)\right)\left(f_4(x) - f_4(a)\right)\right)\right|$ by factoring

$\leq |f_4(x)||f_3(x) - f_3(a)| + |f_3(a)||f_4(x) - f_4(a)|$ by the triangle inequality
$=[(\varepsilon + |f_4(a)|)]\left[\frac{\varepsilon}{2\cdot(\varepsilon+|f_4(a)|)}\right] + |f_3(a)|\left[\frac{\varepsilon}{2\cdot|f_3(a)|+1}\right]$ by substituting
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by simplifying and observing $\frac{|f_3(a)|\varepsilon}{2\cdot|f_3(a)|+1} < \frac{\varepsilon}{2}$.
So we have succeeded in showing that $0 < |x - a| < \delta$ implies $\left|\left(f_3(x) \cdot f_4(x)\right) - \left(f_3(a) \cdot f_4(a)\right)\right| < \varepsilon$ and can say that the product of continuous functions are continuous. With repetition, this means that any polynomial in the form $p(x) = x^n$, where n is a natural number, is continuous. With Lemma 3.4.1, this means that any polynomial in the form $p(x) = c_1x^n$ is continuous. With Lemma 3.4.2, we can finally say $p(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$ is continuous.

Lemma 3.5 Where U is an open set, $f: U \to \mathbb{R}$, and $a \in U$, $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

Pf: First, definitions, the right-hand limit: $\lim_{x\to a^+} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0$ such that whenever $a < x < a + \delta$, it must be the case that $|f(x) - L| < \varepsilon$. Left-hand limit: $\lim_{x\to a^-} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0$ such that whenever $a > x > a - \delta$, it must be the case that $|f(x) - L| < \varepsilon$.

Assume $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$. By the definition of right and left-hand limit, we know that for every positive ε, there are positive values δ_1 and δ_2 such that whenever $x \in (a, a + \delta_1)$, we have $|f(x) - L| < \varepsilon$ and whenever $x \in (a - \delta_2, a)$ we have $|f(x) - L| < \varepsilon$. We want to show that $\lim_{x\to a} f(x) = L$, or that there exists some δ that forces $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. Consider $\delta = min\{\delta_1, \delta_2\}$ and assume $0 <$

$|x-a|<\delta$. By the definition of the absolute value function, $0<|x-a|<\delta$ means that either $a-\delta<x<a$ or $a<x<a+\delta$.

In the first case, we know that $a-\delta_2 \leq a-\delta$ since $\delta_2 \geq \delta$ by construction of δ. As a result, we see $a-\delta_2 \leq a-\delta \ < x < a$, so $a-\delta_2 < x < a$ which implies $|f(x)-L| < \varepsilon$ from the construction of the left-hand limit. In the second case, we know $a+\delta \leq a+\delta_1$ since $\delta_1 \geq \delta$ by the construction of δ. As a result, we see $a<x<a+\delta \leq a+\delta_1$, so $a<x<a+\delta_1$which implies $|f(x)-L|<\varepsilon$ from the construction of the right-hand limit and we have shown $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \Rightarrow \lim_{x\to a} f(x) = L$.

Going the other way, we assume that there exists some $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $x\in(a-\delta,a+\delta)$. But then if $x\in(a-\delta,a+\delta)$, then surely $x\in(a,a+\delta)$. So the δ forces the right-hand limit. Similarly, if $x\in(a-\delta,a+\delta)$, then surely $x\in(a-\delta,a)$. So the δ forces the left-hand limit as well and we have shown $\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

Theorem 3.2: If X $\subset E_1$ is connected and $f: E_1 \to E_2$ is a continuous mapping between metric spaces, then $f(X)$ is connected. That is, continuous functions preserve connectedness.

Pf: Imagine we have the opposite conclusion, that the continuous image of a connected set is disconnected. Then by the definition of disconnectedness there would exists two open and non-empty sets $U,V\subseteq E_2$ whereby $f(X)\subseteq U\cup V$, $U\cap V=\emptyset$, $f(X)\cap U\neq\emptyset$, and $f(X)\cap V\neq\emptyset$.

As f is a continuous function, we can say $f^{-1}(U)$ and $f^{-1}(V)$ are open in E_1 by Theorem 3.1. We can also say the two sets are disjoint. If they shared some common member y, then $y\in f^{-1}(U)\cap f^{-1}(V)$, and so $y\in f^{-1}(U)$ and $y\in f^{-1}(V)$. By the properties of the inverse image, this means that $f(y)\in U$ and $f(y)\in V$, so $f(y)\in U\cap V$, which contradicts the assumption that U and V are disjoint. Lastly we can conclude that $f^{-1}(U)$ and $f^{-1}(V)$ cover X; $X\subseteq f^{-1}(U)\cup f^{-1}(V)$. To show this, we will show that any member of X is in $f^{-1}(U)\cup f^{-1}(V)$. Let $x\in X$ be given. Then $f(x)\in f(X)\subseteq U\cup V$, which means $f(x)\in U$ or $f(x)\in V$, and so $x\in f^{-1}(U)$ or $x\in f^{-1}(V)$ and we can say $x\in f^{-1}(U)\cup f^{-1}(V)$.

Since $f^{-1}(U)$ and $f^{-1}(V)$ are open and disjoint sets that cover X, and X is connected by assumption, it must be the case that $X\cap f^{-1}(U)=\emptyset$ or $X\cap f^{-1}(V)=\emptyset$. But in the first case, this would mean that some generic element x of X would not be in $f^{-1}(U)$, so $f(x)\notin U$. Since no members of the image of X are in U, we would have $f(X)\cap U=\emptyset$, which contradicts the assumption that $f(X)$ is disconnected. Applying the same rationale to the second case, we would have $f(X)\cap V=\emptyset$, again contradicting the assumption that $f(X)$ is disconnected. So X must not be connected and we have reached our contradiction.

4. Differentiability

Maybe the central idea taken from a course in calculus is the idea of a derivative. The wide ranging and intuitive uses for the derivative and its counterpart, the integral, make it an all important topic in applied mathematics. By definition, the derivative of a function f at a point x_0 in the domain is $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, or equivalently $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

The aim of this paper is to prove basic derivative rules using the ideas of open and closedness, continuity, compactness, and connectedness.

Suppose U is an open interval in $\mathbb{R}$, $a, b,$ and c are fixed real numbers, n is a natural number, and $f, g\colon U \to \mathbb{R}$ be differentiable at a point x_0. Then we can show the following results.

1) Derivative of a constant function. If $\forall x \in U, f(x) = c$, then $f'(x_0) = 0$

Pf: $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{c-c}{x-x_0} = \lim_{x \to x_0} \frac{0}{x-x_0}$ which is defined for every value of $x \neq x_0$, and thus we can conclude $f'(x_0) = 0$.

2) Derivative of a power function. Where $\forall x \in U, f(x) = x^n, f'(x_0) = n \cdot {x_0}^{(n-1)}$.

Pf: $f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x-x_0} = \lim_{x \to x_0} \frac{(x-x_0)(x^{n-1}+x^{n-2}x_0+\cdots+x{x_0}^{n-2}+{x_0}^{n-1})}{x-x_0} =$
$\lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + x{x_0}^{n-2} + {x_0}^{n-1})$.

To see this is the case, observe that every term besides the first and last cancel when multiplied.
$(x - x_0)(x^{n-1} + x^{n-2}x_0 + \cdots + x{x_0}^{n-2} + {x_0}^{n-1}) =$
$(x^n - x^{n-1}x_0) + (x^{n-1}x_0 - x^{n-2}{x_0}^2) + \cdots + (x^2{x_0}^{n-2} - x{x_0}^{n-1}) + (x{x_0}^{n-1} - {x_0}^n) =$
$x^n - x_0^n$. Since the function is a polynomial, it is continuous (Lemma 3.3). The definition of continuity necessitates that the limit of a continuous function is simply the function evaluated at the value the limit approaches.
So $\lim_{x \to x_0} (x^{n-1} + x^{n-2}x_0 + \cdots + x{x_0}^{n-2} + {x_0}^{n-1}) =$
${x_0}^{n-1} + {x_0}^{n-2}x_0 + \cdots + x_0{x_0}^{n-2} + {x_0}^{n-1} =$
${x_0}^{n-1} + {x_0}^{n-1} + \cdots + {x_0}^{n-1} + {x_0}^{n-1}$. By counting, we see that there are n terms of ${x_0}^{n-1}$, so we finally have $n \cdot {x_0}^{n-1}$ as required.

3) Derivative of a logarithmic function. Where $\forall x \in U, f(x) = \log_b(x), f'(x_0) = \frac{1}{x_0 \cdot \ln(b)}$.

First, we'll look at the specific case of the derivative of the natural log. By definition, for $f(x) = \ln(x), f'(x) = \lim_{h \to 0} \left[\frac{\ln(x+h)-\ln(x)}{h}\right]$. Recall that $\log_b(x) - \log_b(y) = \log_b\left(\frac{x}{y}\right)$ (see logarithm rule 3 on my previous paper on basic arithmetic). Using this fact, see $\ln(x+h) - \ln(x) = \ln\left(\frac{x+h}{x}\right)$.

Writing the fraction as a product and simplifying, we can write $f'(x) = \lim_{h \to 0} \left[\frac{1}{h} \cdot \ln\left(1 + \frac{h}{x}\right)\right]$.
Creatively multiplying by 1, or x/x, we can write $f'(x) = \lim_{h \to 0} \left[\frac{1}{x} \cdot \frac{x}{h} \cdot \ln\left(1 + \frac{h}{x}\right)\right]$.
Call $n = \frac{x}{h}$ (so $\frac{h}{x} = \frac{1}{n}$, and as $h \to 0, n \to \infty$). Then we can rewrite this as $f'(x) = \lim_{n \to \infty} \left[\frac{1}{x} \cdot n \cdot \ln\left(1 + \frac{1}{n}\right)\right]$. By Lemma 3.1, we are left with $f'(x) = \frac{1}{x} \cdot \lim_{n \to \infty} \left[n \cdot \ln\left(1 + \frac{1}{n}\right)\right]$.
Logarithm rules tell us that $a \cdot \log_b(x) = \log_b(x^a)$ (see logarithm rule 2 in my previous paper on basic arithmetic), so we can write $f'(x) = \frac{1}{x} \cdot \lim_{n \to \infty} \left[\ln\left(\left[1 + \frac{1}{n}\right]^n\right)\right]$. By Lemma 3.3, $f'(x) = \frac{1}{x} \cdot \lim_{n \to \infty} \left[\ln\left(\left[1 + \frac{1}{n}\right]^n\right)\right] = \frac{1}{x} \cdot \ln\left(\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]\right) = \frac{1}{x} \cdot \ln(e) = \frac{1}{x}$ and we have arrived at the derivative of the natural log. We can now generalize the base of the natural logarithm with the change of base formula to find the derivative of any logarithm function. Recall the change of base formula shows $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ (see logarithm rule 3 in my previous paper on basic arithmetic). We want to find the derivative of $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$. So $f'(x_0) = \frac{d}{dx}\left[\ln(x_0) \cdot \frac{1}{\ln(b)}\right]$. This is a variable function multiplied by a constant, so to find the derivative, we can pull out the constant $\frac{1}{\ln(b)}$. We now have $f'(x_0) = \frac{1}{\ln(b)} \cdot \frac{d}{dx}[\ln(x_0)] = \frac{1}{\ln(b)} \cdot \frac{1}{x_0} = \frac{1}{x_0 \cdot \ln(b)}$ and we are done.

4) Derivative of an exponential function. Where $\forall x \in U, f(x) = a^x, f'(x_0) = \ln(a) \cdot a^x$.
Pf: First a specific instance. $\frac{d}{dx}[e^x] = \frac{d}{dx}\left[e^{\mathrm{x} \cdot \ln(a)}\right] = \lim_{h \to 0} \left[\frac{e^{(x+h)} - e^x}{h}\right] = \lim_{h \to 0} \left[\frac{e^x e^h - e^x}{h}\right] = \lim_{h \to 0} \left[\frac{e^x(e^h - 1)}{h}\right]$. Applying Lemma 3.1, we have $e^x \cdot \lim_{h \to 0} \left[\frac{(e^h - 1)}{h}\right]$. Now call $n = e^h - 1$, which means $h = \ln(n + 1)$. Substituting these values, and realizing that as $h \to 0$, $n \to 0$ (any base raised to 0 is 1), we are left with $e^x \cdot \lim_{n \to 0} \left[\frac{n}{\ln(n+1)}\right]$.
Multiplying both the numerator and denominator by $\frac{1}{n}$, we are left with $e^x \cdot \lim_{n \to 0} \left[\frac{1}{\frac{1}{n} \cdot \ln(n+1)}\right]$. By the properties of logarithms discussed in previous papers, we can re-write this as $e^x \cdot \lim_{n \to 0} \left[\frac{1}{\ln\left[(n+1)^{\frac{1}{n}}\right]}\right]$. By Lemma 3.3, we can write $e^x \left[\frac{1}{\ln\left(\lim_{n \to 0}\left[(n+1)^{\frac{1}{n}}\right]\right)}\right]$. Observe that $\lim_{n \to 0} \left[(n + 1)^{\frac{1}{n}}\right]$ is equivalent to $\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]$, which is how e is defined. So we have $\frac{d}{dx}[e^x] = e^x \left[\frac{1}{\ln(e)}\right] = e^x$. Applying the chain rule, for some function $e^{g(x)}$, we know $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$.

Now notice we can write $a^x = e^{\ln(a^x)} = e^{x \cdot \ln(a)}$ by the logarithm rules mentioned in previous papers. So $\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{x\cdot\ln(a)}\right] = e^{x\cdot\ln(a)} \cdot \ln(a)$. Notice the first term in the product is just a^x, so we are left with $\ln(a) \cdot a^x$ as required.

5) Sum Rule. $(f+g)'_{(x_0)} = f'_{(x_0)} + g'_{(x_0)}$.

Pf: $(f+g)'_{(x_0)} = \lim_{x\to x_0}\left[\frac{(f+g)_{(x)}-(f+g)_{(x_0)}}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{\left(f_{(x)}+g_{(x)}\right)-\left(f_{(x_0)}+g_{(x_0)}\right)}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{\left(f_{(x)}-f_{(x_0)}\right)+\left(g_{(x)}+g_{(x_0)}\right)}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{\left(f_{(x)}-f_{(x_0)}\right)}{x-x_0}+\frac{\left(g_{(x)}+g_{(x_0)}\right)}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{\left(f_{(x)}-f_{(x_0)}\right)}{x-x_0}\right] + \lim_{x\to x_0}\left[\frac{\left(g_{(x)}+g_{(x_0)}\right)}{x-x_0}\right] = f'_{(x_0)} + g'_{(x_0)}$. From Lemma 3.2 and the definition of derivative.

6) Product Rule. $(fg)'_{(x_0)} = g_{(x_0)}f'_{(x_0)} + f_{(x_0)}g'_{(x_0)}$.

Pf: $(fg)'_{(x_0)} = \lim_{x\to x_0}\left[\frac{f_{(x)}g_{(x)}-f_{(x_0)}g_{(x_0)}}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{f_{(x)}g_{(x)}+\left(f_{(x_0)}g_{(x)}-f_{(x_0)}g_{(x)}\right)-f_{(x_0)}g_{(x_0)}}{x-x_0}\right] = \lim_{x\to x_0}\left[\frac{f_{(x)}g_{(x)}-f_{(x_0)}g_{(x)}}{x-x_0}+\frac{f_{(x_0)}g_{(x)}-f_{(x_0)}g_{(x_0)}}{x-x_0}\right] = \lim_{x\to x_0}\left[g_{(x_0)}\cdot\frac{f_{(x)}-f_{(x_0)}}{x-x_0}\right] + \lim_{x\to x_0}\left[f_{(x_0)}\cdot\frac{g_{(x)}-g_{(x_0)}}{x-x_0}\right] = g_{(x_0)}f'_{(x_0)} + f_{(x_0)}g'_{(x_0)}$ by Lemma 3.1 and 3.2.

7) Chain Rule. Where U and V are open sets and $f: U \to V$, $G: V \to \mathbb{R}$, and $x_0 \in U$ such that if f is differentiable at x_0, and g is differentiable at $f(x_0)$, then $(g^\circ f)'_{(x_0)} = g'\left(f_{(x_0)}\right) \cdot f'_{(x_0)}$.

Pf: First, define a new function $d: V \to \mathbb{R}$ by $d(y) = \begin{Bmatrix} \frac{g(y)-g\left(f(x_0)\right)}{y-f(x_0)}, y \neq f_{(x_0)} \\ g'\left(f(x_0)\right), y = f_{(x_0)} \end{Bmatrix}$. We know this function is continuous at $y = f(x)$ since $\lim_{y\to f(x)}[d(y)] = \lim_{y\to f(x)}\left[\frac{g(y)-g\left(f(x_0)\right)}{y-f(x_0)}\right] = g'(f(x))$ which exists by the assumption of the proof, and whose value using the limit definition of derivative is $\frac{g(f(x))-g\left(f(x_0)\right)}{f(x)-f(x_0)}$. Notice that for any $y, g(y) - g\left(f(x_0)\right) = d(y) \cdot [f(x) - f(x_0)]$. So for $y = f(x)$ for some $x \in U\backslash\{x_0\}$, we have $g(f(x)\,) - g\left(f(x_0)\right) = d(f(x)) \cdot [f(x) - f(x_0)]$, or $\frac{g(f(x)\,)-g\left(f(x_0)\right)}{x-x_0} = d(f(x)) \cdot \frac{f(x)-f(x_0)}{x-x_0}$ upon dividing by $x - x_0$ on both sides. Taking the limit of both sides, we have $\lim_{x\to x_0}\left[\frac{g(f(x)\,)-g\left(f(x_0)\right)}{x-x_0}\right] = \lim_{x\to x_0}[d(f(x))] \cdot \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]$. So $g'f(x_0) = g'\left(f_{(x_0)}\right) \cdot f'_{(x_0)}$ since d is continuous.

8) Fermat's Extreme Value Theorem: if x_0 is differentiable and a local extrema, $f'(x_0) = 0$.

Pf: Where U is an open set in $\mathbb{R}$ and $f: U \to \mathbb{R}$, $x_0 \in U$ is a local extrema if $\exists \varepsilon > 0$ such that $\forall q \in B_U(x_0, \varepsilon)$, we have either $f(x_0) \leq f(q)$ or $f(x_0) \geq f(q)$. Since x_0 is differentiable, we know $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists (the limit definition from the perspective of the secant line approximation, where h represents the difference between x_0 and x). Further, since x_0 is a local extrema (we will assume it is a local maximum without loss of generality), for sufficiently small h, $f(x_0) \geq f(x_0+h)$. We proceed by cases and note that $f(x_0) \geq f(x_0+h)$ implies $f(x_0+h)-f(x_0) \leq 0$. So if h is positive, $\frac{f(x_0+h)-f(x_0)}{h} \leq 0$. Taking the right-hand limit, we have $\lim_{h\to 0^+}\frac{f(x_0+h)-f(x_0)}{h} \leq \lim_{h\to 0^+} 0 = 0$, and since the derivative at x_0 exists, we know $f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0^+}\frac{f(x_0+h)-f(x_0)}{h} \leq 0$ (by Lemma 3.5). Similarly, if h is negative, $\frac{f(x_0+h)-f(x_0)}{h} \geq 0$. Taking the left-hand limit, we have $\lim_{h\to 0^-}\frac{f(x_0+h)-f(x_0)}{h} \geq \lim_{h\to 0^-} 0 = 0$, and see $f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0^-}\frac{f(x_0+h)-f(x_0)}{h} \geq 0$ and by duality have $f'(x_0) = 0$.

9) Rolle's Theorem: If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$, then there is some $c \in (a,b)$ such that $f'(c) = 0$.

Pf: If a and b are both maximum and minimum values achieved for the function, then we have a constant function. Choose any $c \in (a,b)$, then $f'(c) = 0$ by Theorem 4.1. Otherwise, there is some value $c \in (a,b)$ that is a local extrema, and as a result of Fermat's Extreme Value Theorem can conclude $f'(c) = 0$

10) Mean Value Theorem. If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Pf: This theorem is saying that regardless of the function (so long as it is continuous on the closed interval and differentiable on the open interval), there will be a value in the interval whose tangent line to the curve has the same slope as the secant line through the end points of the function. Define a new function $h: [a,b] \to \mathbb{R}$, $h(x) = f(x) - f(a) - \left[\frac{f(b)-f(a)}{b-a}\cdot(x-a)\right]$ (although not necessary for the proof, we arrive at this value by thinking about the equation of the secant line through the endpoints of f). By Lemma 3.4 and the sum and product rule for derivatives, we know this function is continuous at all points in the domain and differentiable at every point excluding endpoints. Note that $h(a) = f(a) - f(a) - \left[\frac{f(b)-f(a)}{b-a}\cdot(a-a)\right] = 0 - \left[\frac{f(b)-f(a)}{b-a}\cdot(0)\right] = 0$. Further note that $h(b) = f(b) - f(a) - \left[\frac{f(b)-f(a)}{b-a}\cdot(b-a)\right] = f(b) - f(a) - [f(b)-f(a)] = 0$. So we have $h(a) = h(b)$. Rolle's Theorem dictates that $\exists c \in [a,b]$ where $h'(c) = 0$.

Expanding the function, $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}\cdot x + \frac{f(b)-f(a)}{b-a}\cdot a$. Taking the derivative, $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ (note $a, b, f(a)$, and $f(b)$ are all constants). From the above application of Rolle's Theorem, $h'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$, so $f'(c) = \frac{f(b)-f(a)}{b-a}$.

11) Intermediate Value Theorem. If $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then for any value y between $f(a)$ and $f(b)$, there is a $x \in [a, b]$ such that $f(x) = y$.

Pf: Assume that $f(a) \leq f(b)$ (analogous results follow for the reverse assumption). We have shown that $[a, b]$ is connected in $\mathbb{R}$ by Theorem 2.1. Since continuous functions preserve connectedness as shown in Theorem 3.2, we know $f([a, b])$ is connected. But then reapplying Theorem 2.1, we know $f([a, b])$ has the betweeness property and that any y with $f(a) \leq y \leq f(b)$ is in $f([a, b])$. So therefore we know $f^{-1}(y) \in f^{-1}\big(f([a, b])\big) = [a, b]$ and we are done.