1. Formal Definition of Limits and Derivatives

The limit of a function as its inputs tend towards some value a in the domain is some value L in the range if for all $\varepsilon > 0$, there is some value $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it must be the case that $|f(x) - L| < \varepsilon$. So if $\lim_{x \to a} f(x) = L$, whenever a function's input x is in the range $(a - \delta, a + \delta)$, we know that $f(x)$ will be in the range $(L - \varepsilon,\ L + \varepsilon)$.

The limit definition is used in the definition for a derivative. Let $f: \mathbb{R} \to \mathbb{R}$. The derivative of f at some point x_0 in the domain, denoted $f'(x_0)$, is precisely the limit as x approaches x_o of $\frac{f(x)-f(x_0)}{x-x_0}$. As any high school student could tell you, the derivative of a point can be graphically interpreted as the slope of the line tangent to the curve at the given point. What the limit definition says is that the slope of the secant line through two points on the curve tends toward the slope of the tangent line of a point as the two points get arbitrarily close together. Fix some input x and let x_0 vary across the domain. Call h the difference between x and x_0. The slope of the secant line through the points $\big(x, f(x)\big)$ and $\big(x_0, f(x_0)\big)$ will be $\frac{f(x_0+h)-f(x_0)}{h}$, since the slope of a line of a function is the change in outputs over the change in inputs. Then the slope of the tangent line at x will be the limit of the slope of secant line through $\big(x, f(x)\big)$ and $\big(x_0, f(x_0)\big)$ as the distance between x and x_0 become arbitrarily close. Written a different way, $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$, an equivalent statement as previously shown.

We'll now show some useful properties about the limit and derivative that we'll apply later on.

Lemma 1.1, the limit of a constant multiplied by a function is equal to the constant multiplied by the limit of the function.

Pf: Let $\lim_{x \to a} f(x) = L$, and let C be some constant term. We would like to show that $\lim_{x \to a} f(x) = (C \cdot f(x)) = C \cdot \lim_{x \to a} f(x) = C \cdot L$. Since $\lim_{x \to a} f(x) = L$, we know there is some $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it must be the case that $|f(x) - L| < \varepsilon$ for any positive value of ε. Let δ_1 be the value which forces $|f(x) - L| < \frac{\varepsilon}{|C|}$. We want to show the existence of a δ_2 value that forces $|Cf(x) - CL| < \varepsilon$, regardless of how small epsilon is, whenever $0 < |x - a| < \delta_2$. Consider $\delta_2 = \delta_1$. Then when $0 < |x - a| < \delta_2 = \delta_1$, $|Cf(x) - CL| = |C| \cdot |f(x) - L| < |C| \cdot \frac{\varepsilon}{|C|} = \varepsilon$ and we are done. Note that the last inequality is a result of the construction of δ_1.

Lemma 1.2, the limit of the sum is equal to the sum of the limits.

Pf: Let $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$. We'd like to show $\lim_{x \to a}[\, f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L_1 + L_2$. So let $\varepsilon > 0$ be given. We want to specify a value $\delta > 0$ that will ensure $|(f(x) + g(x)) - (L_1 + L_2)| < \varepsilon$ whenever $0 < |x - a| < \delta$. First observe that there is some

$\delta_1 > 0$ that forces $|f(x) - L_1| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_1$ by the definition of a limit. Similarly, there is some $\delta_2 > 0$ that forces $|g(x) - L_2| < \frac{\varepsilon}{2}$ whenever $0 < |x - a| < \delta_2$. So select $\delta = \max\{\delta_1, \delta_2\}$.

We know $|(f(x) + g(x)) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)|$, which by the property of absolute value is less than or equal to $|f(x) - L_1| + |g(x) - L_2|$. By the construction of δ, $|f(x) - L_1| < \frac{\varepsilon}{2}$ and $|g(x) - L_2| < \frac{\varepsilon}{2}$, so $|(f(x) + g(x)) - (L_1 + L_2)| < \varepsilon$ as required.

Lemma 1.3, derivatives under arithmetic.

Pf: We'd like to demonstrate that for a differentiable function $f(x)$, the derivative of $c \cdot f(x)$ (where c is a constant) is $c \cdot f'(x)$. To do so, we'll use Lemma 1.1.

By the definition of derivative, $[c \cdot f(x)]' = \lim_{x \to x_0} \frac{c \cdot f(x) - c \cdot f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c \cdot (f(x) - f(x_0))}{x - x_0}$. Lemma 1.1 showed that we can pull out constants, so $[c \cdot f(x)]' = c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c \cdot f'(x)$ and we are done.

2. Geometry of a Triangle and Trigonometric Functions

Select three distinct points in two-space that don't lie on a line. The shape formed after extending straight lines that connect each of the points (called vertices) is a triangle. Label each of the three sides of the triangle A, B, and C respectively. Then the internal angle between sides A and B (or sides A and C) is measured in degrees (where one degree is 1/360th of a full rotation of the fixed length about a vertex) or in radians (which measures the arc length on the unit circle formed by the rotation of side A to side B). It is convenient to associate one pi radian with a 180° rotation about a vertex. We will take it as an axiom that the sum of the internal angles of a triangle is 180 degrees, and will also assume that the reader is familiar with the conventions around angle measurements.

A right-triangle is a special type of triangle with one internal angle of precisely 90-degrees; it's a triangle with two perpendicular sides. One can classify the sides of a right triangle based on an observed internal angle. Fix some internal angle θ (the other internal angles will be $\frac{\pi}{2}$ and $\frac{\pi}{2} - \theta$). Then the side of the triangle independent of the formation of the right-angle is called the hypotenuse, the side of the triangle which forms the internal angle θ with the hypotenuse is called the adjacent, and the remaining side is called the opposite.

Trigonometric functions give the ratio of side lengths on any right-triangle based on an observed internal angle. The two principal trigonometric functions are sine and cosine. The sine of an angle measurement gives the ratio of the opposite side to hypotenuse. The cosine of the same angle gives the ratio of the adjacent length to the hypotenuse length. Note that when the hypotenuse of a right-triangle has unit length (like when a triangle is inscribed within the unit circle and the adjacent side lies along the x-axis) the cosine of the angle will be the adjacent length, and the sine of the angle will be the opposite length.

Since a triangle has three distinct sides, there are 6 unique trig functions. They all follow from the sine and cosine values of an angle. The remaining functions are called the tangent (the ratio of an opposite to an adjacent, also the ratio of the sine of an angle to the cosine of an angle), the secant (the inverse of cosine, the ratio of the hypotenuse to adjacent), the cosecant (the inverse of sine, the ratio of the hypotenuse to the opposite), and the cotangent (the inverse of tangent, the ratio of an adjacent to the opposite). While trig functions have unrestricted domains, we'll only focus on those inputs in the interval $\left[0, \frac{\pi}{2}\right]$ since we're looking at the intuition behind the functions within the framework of a right-triangle.

For thoroughness though, we give the specific values of common degree measurements:

Degree	0	30	45	60	90	120	135	150	180
Radian	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
Sine	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
Cosine	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Observe that the range of the sine and cosine function falls in the closed interval $[-1,1]$. Also note that the sine function is cyclical with a period of 2π and roots at every natural multiple of π. Similarly, the cosine function has roots at every value $\frac{k\pi+1}{2}$ where k is a natural number.

Let's now look at the two specific properties of trigonometric functions.

Lemma 2.1, the sine angle-addition formula, that $\sin(a+b) = \cos a \cdot \sin b + \cos b \cdot \sin a$

Pf: This proof will be geometric and utilize the fact that alternate angles on a line that intersects two perpendicular lines are equal. Fix a length $\overline{AB}$. Rotate this length α degrees counter-clockwise from the point A, and raise a line perpendicular to the line $\overline{AB}$ with length $\overline{BC}$. So we have a right-triangle ΔABC that has a hypotenuse of $\overline{AC}$, opposite of $\overline{BC}$, and adjacent of $\overline{AB}$ with angle measure α. Select a point D between points A and B, and create a line perpendicular to $\overline{AB}$ with some length $\overline{ED}$. Then connect points A and E. We now have a second right-triangle ΔADE with a hypotenuse of $\overline{AE}$, opposite of $\overline{ED}$, and adjacent of $\overline{AD}$ with some angle measure θ. Let $\beta = \theta - \alpha$ and call the point that lies on both $\overline{AC}$ and $\overline{ED}$ the point F. Next, draw a line connecting E and C and notice that we have a third and fourth right-triangle, namely ΔFCE and ΔACE. To finish the set-up, draw a line from the point C to the line $\overline{ED}$ that runs parallel to $\overline{AB}$, and label the point on $\overline{ED}$ point G. We now have two additional right-triangles: ΔFGC and ΔCEG.

Next, we realize that the angle formed by lines $\overline{GC}$ and $\overline{CF}$ will be α (since $\overline{AC}$ intersects the parallel lines $\overline{AB}$ and $\overline{GC}$, so the angle is alternate the angle formed by the lines $\overline{AB}$ and $\overline{AC}$). Triangle ΔACE has a right-angle formed by lines $\overline{AC}$ and $\overline{CE}$, so the angle formed by the lines $\overline{GC}$ and $\overline{CE}$ will be $\frac{\pi}{2} - \alpha$. Finally, since ΔCEG is a right-triangle, we can conclude that the angle formed by lines $\overline{EG}$ and $\overline{EC}$ will be α.

We are trying to determine a formula for the sine of the sum of two angles. Observe that $\sin(\theta) = \sin(\alpha + \beta)$ will be the ratio of $\overline{ED}$ to $\overline{EA}$. Further observe that $\overline{ED} = \overline{EG} + \overline{BC}$, so this ratio can be re-written as $\overline{EG}/\overline{EA} + \overline{BC}/\overline{EA}$. Multiply the first ratio by $\overline{CE}/\overline{CE}$ and the second ratio by $\overline{AC}/\overline{AC}$, we get $\sin(\theta) = \sin(\alpha + \beta) = \frac{\overline{EG}}{\overline{CE}} \cdot \frac{\overline{CE}}{\overline{EA}} + \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{EA}}$. Going term by term, $\frac{\overline{EG}}{\overline{CE}}$ is the ratio of the adjacent to the hypotenuse in ΔGEC (so its $\cos\alpha$), $\frac{\overline{CE}}{\overline{EA}}$ is the ratio of opposite to the hypotenuse in ΔACE (so its $\sin\beta$), $\frac{\overline{BC}}{\overline{AC}}$ is the ratio of the opposite to the hypotenuse in ΔABC (so its $\sin\alpha$), and $\frac{\overline{AC}}{\overline{EA}}$ is the ratio of the adjacent to the hypotenuse in ΔACE (so its $\cos\beta$). Reordering terms, we see $\sin(\theta) = \sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$ as required.

Lemma 2.2, the trigonometric identity $sin^2\theta + cos^2\theta = 1$.

Pf: This identity is a result of the Pythagorean Theorem. Form a right triangle with a hypotenuse of length C, opposite of length B, and adjacent of length A. The Pythagorean Theorem tells us that $A^2 + B^2 = C^2$, and upon by dividing both sides by C^2, $\frac{A^2}{C^2} + \frac{B^2}{C^2} = 1$. Notice that the first term is the square of the ratio of the opposite to the hypotenuse and the second term is the square of the ratio of the adjacent to the hypotenuse.

Lemma 2.3, the cosine angle-addition formula, that $\cos(a + b) = \cos a \cdot \cos b - \sin a \cdot \sin b$

Pf: Create some rectangle in two-space with points $A, B, C,$ and D such that a line can be drawn from A to a point on the line $\overline{CD}$, call it E, with length 1. Next, identify a point F on the line $\overline{BC}$ wherein there is a right-angle formed between lines $\overline{EF}$ and $\overline{AE}$. We now have four separate right-triangles. Call the angle between lines $\overline{AB}$ and $\overline{AF}$ α and the angle between lines $\overline{AB}$ and $\overline{AF}$ β. Observe that the angle between lines $\overline{DE}$ and $\overline{AE}$ will be $\alpha + \beta$ (since $\overline{AE}$ intersects the parallel lines $\overline{AB}$ and $\overline{CD}$ making it the angle opposite that formed by lines $\overline{AB}$ and $\overline{AE}$). Also note that the angle between lines $\overline{EF}$ and $\overline{CF}$ will be α since straight lines have π degrees, and the angle formed by lines $\overline{BF}$ and $\overline{AF}$ is $\frac{\pi}{2} - \alpha$ while the angle formed by lines $\overline{EF}$ and $\overline{AF}$ is $\frac{\pi}{2}$.

With this set-up in mind, we can calculate the length of other line segments. Reformatting the equation for sine and cosine, we see that the length of $\overline{AF}$ is $\cos\beta$ while the length of $\overline{EF}$ is $\sin\beta$ in triangle ΔAEF. Then in triangle ΔEFC, we see that $\sin\alpha = \frac{\overline{EC}}{\overline{EF}} = \frac{\overline{EC}}{\sin\beta}$, so $\overline{EC} = \sin\alpha \sin\beta$.

Using the same methodology in triangle ΔABF, we see that $\cos\alpha = \frac{\overline{AB}}{\overline{AF}} = \frac{\overline{AB}}{\cos\beta}$, so $\overline{AB} = \cos\alpha\cos\beta$. Now looking at triangle ΔEAD, the length of $\overline{ED}$ will be $\cos(\alpha+\beta)$ since the hypotenuse has length 1. Using our past results, $\overline{ED} = \overline{AB} - \overline{CE} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ as required.

3. Common Transcendental Numbers

Mathematicians are able to group numbers based on their characteristics. Without getting too much into set theory, every natural number (counting numbers like 1, 2, and 3), denoted $\mathbb{N}$, is a subset of the integers $(\dots, -2, -1, 0, 1, 2, \dots)$, denoted $\mathbb{Z}$, which is a subset of the rational numbers (numbers able to be written in the form $^{a}/_{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$ such as $^{-3}/_{2}$), denoted $\mathbb{Q}$, which is a subset of the real numbers (any number that can be expressed as an infinite decimal expansion, such as π), denoted $\mathbb{R}$, which is a subset of the complex numbers (numbers in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$), denoted $\mathbb{C}$. We can write $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, and realize that every natural number is a complex number, but not every complex number is a natural number, etc.

Real numbers that are not rational are appropriately called irrational numbers and are denoted $\mathbb{P}$. The union of the rational and irrational numbers form the real numbers, $\mathbb{Q} \cup \mathbb{P} = \mathbb{R}$. Going further, some irrational numbers are identified as transcendental. This means that they are not algebraic—they are not the root of any non-zero polynomial with rational coefficients. For example's sake, every non-perfect square is an irrational number, yet they are all algebraic. Take $\sqrt{2}$, which is irrational (much to the chagrin of the Pythagorean Brotherhood!) but is the root of the polynomial $x^2 - 2 = 0$ and thus algebraic. While we know that most real numbers are transcendental, it's difficult to actually state such numbers. Euler's number, pi, and the golden ratio φ are the premier examples of transcendental numbers.

Euler's Number and Pi

Euler's number e is defined as the unique limit of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ as the inputs tend towards infinity. It has a value of approximately 2.71828.

There are many interesting properties of e. For one, any power k of e is equal to the derivative of e^k. This also implies that the area under the curve e^k is equal to the value of e^k. While this may excite people like me, the real value of e is its use in compounding. The formation of the number should sufficiently explain this.

The other well-known transcendental is pi, the ratio of any circle's circumference to its diameter. For a rather funny debate about which number is "better", take a peek at the five videos in the series here: https://www.youtube.com/watch?v=whpAX30vjoE.

4. Taylor Polynomial Approximations and Intuition behind Maclaurin Series

Let f be some real-valued function where we know the value of $f(a)$. We would like to approximate this function at values of x near a by defining another function. One technique would be to base our approximations on a linear polynomial, call it P_0, that takes the value $f(a)$ for all values x in the domain. This would ensure that f and P_0 have the same value when $x = a$, but would likely have wildly inaccurate estimates for values of $x \neq a$. Take the function $f(x) = x^4$. Of course, $f(1) = 1$. If we use the function $P_0(x) = 1$ to approximate the values of $f(x)$, then we would get very poor estimates. For example when $x = 2, f(2) \approx P_0(2) = 1 \neq f(2) = 16$.

A better approximating function, call it P_1, would still ensure that $P_1(a) = f(a)$ and would additionally utilize the slope of the function at a to estimate values near a. So such a function would necessitate the first-order derivatives of P_1 and f at a to be equal. Then define $P_1(x) = f(a) + f'(a)(x - a)$. Observe that $P_1(a) = f(a) + f'(a)(a - a) = f(a)$ as required. Further, observe that $P_1{}'(x) = f'(a)$ since $P_1(x) = f(a) + f'(a)x - f'(a)a$ and $f(a), f'(a)$, and a are just constants. So $P_1{}'(a) = f'(a)$. This will be a better approximation than P_0 since any value of x close to a will be approximated by the tangent line through $f(a)$. In other words, approximations for $f(x)$ will be based on the value $f(a)$ plus some term that approximates how outputs change across the tangent line based on the difference in inputs. Take the same function as before, $f(x) = x^4$, and see that $f'(x) = 4x^3$, so $f'(1) = 4$. Also, keep the same known value, $f(1) = 1$. Then $P_1(x) = 1 + 4(x - 1) = 4x - 3$, and $f(2) \approx P_1(2) = 5 \neq 16$. This is still a poor approximation, but it's an improvement from P_0.

We can do even better. Imagine a function P_3 that ensures $P_3(a) = f(a)$ and that the first, second, and third derivatives of P_1 and f at a are equal. Such a function would be defined as $P_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{6}(x - a)^3$. Observe that $P_3(x) = f(a)$. Going term by term with the first derivative, we see the following:

- $\frac{d}{dx}[f(a)] = 0$
- $\frac{d}{dx}[f'(a)(x - a)] = \frac{d}{dx}[f'(a)x - f'(a)a] = f'(a)$

- $\frac{d}{dx}\left[\frac{f''(a)}{2}(x-a)^2\right] = \frac{d}{dx}\left[\frac{f''(a)}{2}(x^2-2ax+a^2)\right] = \frac{d}{dx}\left[\frac{f''(a)}{2}\cdot x^2 - af''(a)\cdot x + \frac{a^2\cdot f''(a)}{2}\right] = f''(a)\cdot x - af''(a)$
- $\frac{d}{dx}\left[\frac{f'''(a)}{6}(x-a)^3\right] = \frac{d}{dx}\left[\frac{f'''(a)}{6}(x^3-3x^2a+3xa^2-a^3)\right] = \frac{d}{dx}\left[\frac{f'''(a)}{6}\cdot x^3 - \frac{a\cdot f'''(a)}{2}\cdot x^2 + \frac{a^2\cdot f'''(a)}{2}\cdot x - \frac{a^3\cdot f'''(a)}{6}\right] = \frac{f'''(a)}{2}\cdot x^2 - af'''(a)\cdot x + \frac{a^2\cdot f'''(a)}{2}$.

So $P_3'(x) = f'(a) + f''(a)\cdot x - f''(a)a + \frac{f'''(a)}{2}\cdot x^2 - af'''(a)\cdot x + \frac{a^2\cdot f'''(a)}{2}$. Grouping terms, $P_3'(x) = f'(a) + [f''(a)(x-a)] + \left[f'''(a)\left(\frac{x^2}{2} - ax + \frac{a^2}{2}\right)\right]$. Finally, we can see $P_3'(a) = f'(a) + [f''(a)(a-a)] + \left[f'''(a)\left(\frac{a^2}{2} - a^2 + \frac{a^2}{2}\right)\right] = f'(a)$. We can perform similar operations to show the second-derivative. Note that $P_3''(x) = \left(P_3'(x)\right)'$. Again going term by term, we see: $\frac{d}{dx}[f'(a)] = 0, \frac{d}{dx}[f''(a)(x-a)] = f''(a)$, and $\frac{d}{dx}\left[f'''(a)\left(\frac{x^2}{2} - ax + \frac{a^2}{2}\right)\right] = \frac{d}{dx}\left[\frac{f'''(a)}{2}\cdot x^2 - af'''(a)\cdot x + \frac{a^2 f'''(a)}{2}\right] = f'''(a)x - af'''(a)$. Then $P_3''(x) = f''(a) + f'''(a)x - af'''(a) = f''(a) + f'''(a)(x-a)$. So $P_3''(x) = f''(a) + f'''(a)(a-a) = f''(a)$. At last, we can look at the third-derivative. Going term by term one more time: $\frac{d}{dx}[f''(a)] = 0$, and $\frac{d}{dx}[f'''(a)(x-a)] = f'''(a)$. So $P_3'''(x) = f'''(a)$ and $P_3'''(a) = f'''(a)$. We have shown that P_3 has the same value, first-derivative, second-derivative, and third-derivative as f at the point a. The approximation for our example function at the point $x = 2$ is now much improved. Note that $f'(x) = 4x^3, f''(x) = 12x^2$, and $f'''(x) = 24x$, so $f'(1) = 4, f''(1) = 12$, and $f'''(1) = 24$. Then $P_3(x) = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3$, and $P_3(2) = 1 + 4 + 6 + 4 = 15$. See that $P_3(2)$ is a much better approximation of $f(2)$ relative to either P_1 or P_0.

As one would expect, the more terms in the polynomial, the better the approximation. As such, the infinite Taylor Polynomial approximation of a function is $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$, this is called the Taylor Series. If we choose to center the function at the point $a = 0$, the Taylor Series is called the Maclaurin Series.

Maclaurin Series representation for e^{ix}

First, we show the Maclaurin series for $f(y) = e^y$. We know $f(0) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(y)^n$, or better yet $\sum_{n=0}^{\infty}\frac{e^{0^{(n)}}y^n}{n!} = \frac{y^n}{n!}$. Substituting ix for y, we see $e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!}$. Expanding, $e^{ix} = \left[\frac{(ix)^0}{0!}\right] + \left[\frac{(ix)^1}{1!}\right] + \left[\frac{(ix)^2}{2!}\right] + \left[\frac{(ix)^3}{3!}\right] + \cdots$. Recall that $i^2 = -1$, so we can simplify the numerator in many of the terms. Specifically, observe that $(ix)^{4k} = x^{4k}, (ix)^{4k+1} = x^{4k+1}, (ix)^{4k+2} = -x^{4k+2}$, and $(ix)^{4k+3} = -ix^{4k+3}$ regardless of the natural number k that is chosen. So the infinite sum can be rewritten as $e^{ix} = [1] + [ix] + \left[\frac{-(x^2)}{2!}\right] + \left[\frac{-(ix)^3}{3!}\right] + \left[\frac{(x)^4}{4!}\right] + \cdots$ If we group every term with a coefficient of i together, we have $e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$

5. Using Maclaurin Series for Sine and Cosine to Prove Euler's Formula

Derivative of Sine and Cosine

Before getting to the Maclaurin series for the trig functions outlined above, we must derive the value of the derivative of the sine and cosine functions. Using the limit definition of a derivative, when $f(x) = sinx, f'(x) = \lim_{h\to 0} \frac{sin(x+h)-sin(x)}{h}$. We can re-write this using the angle-addition formula shown in Lemma 2.1: $f'(x) = \lim_{h\to 0} \frac{(\cos x \cdot \sin h + \sin x \cdot \cos h) - \sin x}{h}$. Rearranging, $f'(x) = \lim_{h\to 0}\left(\frac{\cos x \cdot \sin h}{h} + \frac{\sin x \cdot \cos h - \sin x}{h}\right)$. Lemma 1.2 show us that the limit of the sum is equal to the sum of the limits, $f'(x) = \lim_{h\to 0}\left(\frac{\cos x \cdot \sin h}{h}\right) + \lim_{h\to 0}\left(\frac{\sin x \cdot \cos h - \sin x}{h}\right)$. Lemma 1.1 shows that one can pull out coefficients. So taking $\cos x$ from the first limit and $-\sin x$ from the second, we are left with $f'(x) = \cos x \lim_{h\to 0}\left(\frac{\sin h}{h}\right) - \sin x \lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)$. We assert that the first limit is 1, and the second is 0, so the derivative of the sine function is the cosine function.

For the first case, let h be the angle of a right-triangle whose hypotenuse is completely contained (disregarding the endpoint) within the unit circle. Call this triangle A. Form another right-triangle with the same angle, this time with an adjacent length equal to the radius. Call this triangle B. Note that triangle A has a base of $\cos h$ and a height of $\sin h$, so the area of triangle A is $\frac{cos\, h\ \ sin\, h}{2}$. Triangle B has a base length of 1, and a height of $\tan h$ (recall the tangent is the ratio of the opposite to the adjacent, and that the adjacent length is 1). This makes the area of triangle B $\frac{tan\, h}{2}$. Further, the area of the section of the unit circle formed with arc length h is by definition $h/2$. If we label the diameter of the circle d, the radius r, and the circumference c, $c = \pi d = 2\pi$ (rearranging the equation for pi, and using the fact that the diameter of the unit circle is 2), and the area of the whole circle is $\pi r^2 = \pi(1)^2 = \pi$. Since we are taking h/c portions of the circle's whole area, the section has area $\frac{h}{2\pi} \cdot \frac{\pi}{1} = \frac{h}{2}$. So we know $\frac{cos\, h\ \ sin\, h}{2} \leq \frac{h}{2} \leq \frac{tan\, h}{2}$. Multiplying each part of the inequality by $2/\sin h$, we get $\cos h \leq \frac{h}{\sin h} \leq \frac{tan\, h}{\sin h} = \frac{1}{\cos h}$, and after taking the reciprocal, we see that $\frac{1}{cos\, h} \geq \frac{\sin h}{h} \geq \cos h$. So as h tends to zero, we know $\frac{1}{cos\, h} = 1 \geq \frac{\sin h}{h} \geq \cos h = 1$, so $\lim_{h\to 0} \frac{\sin h}{h} = 1$ by the squeeze theorem.

For the second case, we multiply the fraction by $\frac{1+\cos h}{1+\cos h}$. We are left with $\lim_{h\to 0} \frac{1-\cos^2 h}{h\,(1+\cos h)}$. Using the trig identity shown in Lemma 2.2, this is equivalent to $\lim_{h\to 0} \frac{sin^2 h}{h\,(1+\cos h)} = \lim_{h\to 0} \frac{\sin h}{h} \frac{\sin h}{(1+\cos h)}$. We have shown that the first factor tends to 1, and observe that the second factor tends to 0 since 0 is in the numerator. We have now shown $\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right) = 0$, and can conclude that the derivative of the sine function is cosine. This is a pretty spectacular observation if one thinks about it.

Next, we show that the derivative of the cosine function is negative sine. There is a creative argument for this. We've already shown that the values of cosine lag values of sine by $\pi/2$. That is to say that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. Since $[\sin x]' = \cos x$, we also know that $\left[\sin\left(x + \frac{\pi}{2}\right)\right]' = \cos\left(x + \frac{\pi}{2}\right)$. By substitution, this means $[\cos x]' = \cos\left(x + \frac{\pi}{2}\right)$. From Lemma 2.3, we know $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cdot \cos\frac{\pi}{2} - \sin x \cdot \sin\frac{\pi}{2} = -\sin x$. With this information in mind, we can tackle the Maclaurin Series for each infinitely differentiable function.

Maclaurin Series representation for sin and cos

First note that $\forall\, k \in \mathbb{N}$, we have $\frac{d^{4k+0}}{dx^{4k+0}}\,[\sin(x)] = \sin(x)$, $\frac{d^{4k+1}}{dx^{4k+1}}\,[\sin(x)] = cos(x)$, $\frac{d^{4k+2}}{dx^{4k+2}}\,[\sin(x)] = -sin(x)$, and $\frac{d^{4k+3}}{dx^{4k+3}}\,[\sin(x)] = -cos(x)$. This directly follows from Lemma 1.3. Similarly, we see $\frac{d^{4k+0}}{dx^{4k+0}}\,[\cos(x)] = \cos(x)$, $\frac{d^{4k+1}}{dx^{4k+1}}\,[\cos(x)] = -sin(x)$, $\frac{d^{4k+2}}{dx^{4k+2}}\,[\cos(x)] = -cos(x)$, and $\frac{d^{4k+3}}{dx^{4k+3}}\,[\cos(x)] = sin(x)$.

If we choose to center the function $f(x) = \sin(x)$ at zero and use the Maclaurin series representation, we have $f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{sin^{(n)}(0)}{n!}(x)^n$. Expanding, we have $\sin(x) = \left[\frac{sin(0)}{0!}(x)^0\right] + \left[\frac{cos(0)}{1!}(x)^1\right] + \left[\frac{-sin(0)}{2!}(x)^2\right] + \left[\frac{-cos(0)}{3!}(x)^3\right] + \cdots$ Recall that $\sin(0) = 0$, so we can ignore every term with sin in the numerator. Grouping the remaining terms, $\sin(x) = + \left[\frac{cos(0)}{1!}(x)^1\right] + \left[\frac{-cos(0)}{3!}(x)^3\right] + \left[\frac{cos(0)}{5!}(x)^5\right]$ ...Recall that $\cos(0) = 1$, so we can simplify this expansion further. Ultimately, we have $\sin(x) = [x] + \left[\frac{-x^3}{3!}\right] + \left[\frac{x^5}{5!}\right] + \left[\frac{-x^7}{7!}\right] + \cdots$

Now looking at the function $g(x) = cos(x)$ at zero, we have $g(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{cos^{(n)}(0)}{n!}(x)^n = \left[\frac{cos(0)}{0!}(x)^0\right] + \left[\frac{-sin(0)}{1!}(x)^1\right] + \left[\frac{-cos(0)}{2!}(x)^2\right] + \left[\frac{sin(0)}{3!}(x)^3\right] + \cdots = [1] + \left[\frac{-x^2}{2!}\right] + \left[\frac{x^4}{4!}\right] + \left[\frac{-x^6}{6!}\right] + \cdots$

Euler's Formula and Euler's Identity

The Maclaurin Series for e^{ix} is $\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)$ as shown in section 4. Notice that the first grouping is the exact Maclaurin Series for $cos(x)$ and the second grouping is the Maclaurin Series for $\sin(x)$. So we can write $e^{ix}=\cos(x)+i\cdot\sin(x)$. This is Euler's Formula, which is remarkable for many reasons but maybe most abruptly for the way it showcases a simplistic relationship between imaginary powers of a transcendental number and the basic trigonometric functions, two seemingly unrelated concepts.

Once we have arrived at Euler's formula, we can substitute $\pi=x$ and revel in the fact that $e^{i\pi}=\cos(\pi)+i\cdot\sin(\pi)$. It is an elementary fact that the cosine of odd amounts of pi radians is the value -1, and the sine of odd amounts of pi radians is the value 0. So finally we can write Euler's Identity, one of the most beautiful formulas in mathematics: $e^{i\pi}=-1$ or $e^{i\pi}+1=0$!