Exponent Rules: Let $a, b, n,$ and m be real numbers. Then we say $a^n = b$ if n multiples of a is equal to the value b. For a positive value of n, $a^{-n} = \frac{1}{a^n}$ (n divisors of a). We call a the base, and n the exponent. From this definition, we see $a^1 = a$. Without much additional work, we can deduce some more laws of exponentiation.

1) $a^{(m+n)} = a^m a^n$, and $a^{(m-n)} = \frac{a^m}{a^n}\ \forall m, n \in \mathbb{R}$.
   Pf: a^{m+n} is equivalent to a multiplied by itself $(m+n)$ times. Since multiplication is commutative, we can group these $m+n$ terms how we'd like. So we can have m multiples of a multiplied by n multiples of a, which is precisely what we wanted to show. As a consequence, we note that $a^{(m-n)} = a^{(m+(-n))} = a^m a^{-n} = a^m \frac{1}{a^n} = \frac{a^m}{a^n}$
2) $a^0 = 1$.
   Pf: From the first rule, we know that $a^{m+n} = a^m a^n\ \forall m, n \in \mathbb{R}$. Then $a^{m+0} = a^m a^0$. Solving for a^0, we have $a^0 = \frac{a^{m+0}}{a^m} = 1$.
3) $a^{(mn)} = (a^m)^n$, and $a^{(m/n)} = (a^m)^{1/n}\ \forall m, n \in \mathbb{R}, n \neq 0$.
   Pf: $(a^n)^m$ is equivalent to m multiples of a^n. Each a^n has n multiples of a, and there are m of these a^n's, so by multiplying terms, we have $m \cdot n$ multiples of a.
4) $(ab)^n = a^n b^n$, and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
   Pf: Be careful with applying this to structures that aren't commutative!
   $(ab)^n$ is equivalent to n multiples of ab. Since multiplication is commutative, this is the same thing as n multiples of a multiplied by n multiples of b. As a consequence, we see that $\left(\frac{a}{b}\right)^n = \left(a \cdot \frac{1}{b}\right)^n = a^n \frac{1}{b^n} = \frac{a^n}{b^n}$ and are done.

Logarithm Rules: Let $a, b,$ and n be real numbers. Then the logarithm function can be denoted $\log_b x = n$ ("log base b of n") if $b^n = x$. It should be clear that the domain of a logarithmic function is restricted to the positive non-zero reals (only a base of 0 can multiply by itself n times to arrive at $x = 0$, but it does so for any n; only a negative base can multiply by itself n times to arrive at $x < 0$, but then would be undefined for many values). By restricting the domain, we also restrict the base for the same rationale.

1) $\log_b(x \cdot y) = \log_b x + \log_b y$, and $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
   Pf: Let $\log_b x = m_1$ and $\log_b y = m_2$, so that $b^{m_1} = x$ and $b^{m_2} = y$.
   For the first statement, see that $x \cdot y = b^{m_1} \cdot b^{m_2} = b^{(m_1+m_1)}$ from the first exponent rule. We are looking for the n such that $b^n = x \cdot y = b^{(m_1+m_1)}$, which is just $m_1 + m_2$ and we are done.

   For the second statement, see that $\frac{x}{y} = \frac{b^{m_1}}{b^{m_2}} = b^{(m_1-m_2)}$ from the first exponent rule. We are looking for the n such that $b^n = \frac{x}{y} = b^{(m_1-m_1)}$, which is just $m_1 - m_2$ and we are done.
2) $\log_b(x^y) = y \cdot \log_b x$.
   Pf: Call $\log_b x = m_1$, so $b^{m_1} = x$. Then $x^y = (b^{m_1})^y = b^{m_1 y}$ from the third exponent rule. We are looking for the value n such that $b^n = x^y = b^{m_1 y}$, which is just $m_1 y = y \cdot \log_b x$ and we are done.
3) $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$. The change of base formula.
   Pf: Call $\log_b(x) = n$ so $b^n = x$. Similarly, call $\log_a(x) = m_1$ and $\log_a(b) = m_2$ so that $a^{m_1} = x$ and $a^{m_2} = b$. Substituting these values into the first equation, we have $(a^{m_2})^n = a^{m_1}$. From the third exponent rule, this is equivalent to $a^{m_2 n} = a^{m_1}$, or $\frac{a^{m_2 n}}{a^{m_1}} = 1$. From the first exponent rule, this is equivalent to $a^{(m_2 n - m_1)} = 1$. From the second exponent rule, we know that $a^0 = 1$, so have $m_2 n - m_1 = 0$, or $n = \frac{m_1}{m_2}$. Resubstituting for m_1 and m_2, we achieve our result.

Common and Natural Log

There are two logarithmic functions that deserve singling out for the prevalence of their usage. The common log is that function with base 10. The natural log is that function with base e, Euler's Number, and is denoted ln.

Since it's standard to operate in base 10, the prevalence of the common log is unsurprising. To understand why the natural log is such an important function, one must first understand why e is so special.

Similar to how π links all circles together as the ratio of their circumference to their diameter, and the basis of a vector space scales to create any vector in the space, e links all continuously compounded growth or decay processes. More specifically, e is the amount by which an original amount will grow or decay if one continuously compounds 100% growth/unit period. It should be clear that the scaling factor of 100% growth is formulated as $\left(1+\frac{1}{n}\right)^n$, where n is the number of times the growth is compounded in a unit period. Since e represents continuous compounding, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718$. This is admittedly a smaller quantity than one would likely expect. So, for example, $a \cdot e^x$ represents the amount that an initial value of a will grow into if it has a growth rate of r/unit period, compounded continuously for n unit periods, where $x = r \cdot n$.

From this perspective, the natural log function $\ln(x) = y$ gives the multiple of growth rate and time y necessary for a continuously compounded unit value to reach x.

Classifying Common Single-Variable, Real-Valued Functions
For a function $f: \mathbb{R} \to \mathbb{R}, x \to f(x)$, where $a, b, c, d \in \mathbb{R}$ and $n \in \mathbb{N}, n \neq 0$:

1) Constant Function: $f(x) = c$
   A horizontal line.
2) Linear (Affine) Function: $f(x) = a \cdot x + c$
   A straight line whose angle relative to the x-axis is determined by a, and whose y-intercept, or more descriptively vertical shift relative to the origin, is determined by c.
3) Power Function: $f(x) = a \cdot x^n + c$
   A curved line that is shaped like a parabola for even values of n, and "S" shaped for odd values of n. The width of the graph is determined by a. For even exponents, large positive values of a create a thin upward opening parabolic shape and large negative values of a have a thin downward opening parabolic shape. For odd exponents, large positive values of a create a thin "S" shape while large negative values of a create a thin backwards "S" shape. c reflects the shift of the function from the origin.
4) Polynomial Function: $f(x) = a_{n_1} \cdot x^{n_1} + \cdots + a_{n_2} \cdot x^{n_2} + \cdots + c$
   The first term determines the general shape of the function. The highest value of n is called the degree of the polynomial.
5) Exponential Function: $f(x) = a \cdot n^{(bx+c_1)} + c_2$
   Characterized by rapid growth or decay. The value of b determines the width of the function—large magnitudes of b result in a function whose growth/decay near 0 is more extreme than lesser magnitudes of b. Values of c_1 translate the graph c_1 units to the left, while values of c_2 translate the graph c_2 units up. Simply changing the value of b to $-b$ results in a reflection of the function across the y-axis. Simply changing the value of a to $-a$ results in a reflection of the function across the x-axis. Values of a determine whether the function is experiencing exponential growth or decay. A positive a and b results in a graph that is exponentially increasing from a horizontal asymptote of c_2. Positive values of a and negative values of b results in a graph that is exponentially decreasing from a value of positive infinity to a horizontal asymptote of c_2. Negative values of a and positive values of b results in a graph that is exponentially decreasing from a horizontal asymptote of c_2. Negative values of both a and b results in a graph that is exponentially increasing from negative infinity to a horizontal asymptote of c_2.
6) Logarithmic Function: $f(x) = a \cdot \log_b(nx + c) + d$
   The behavior of the logarithm function is of course inverse to that of the exponential function. As previously discussed, the values of the function are only defined for when $nx + c > 0$ and $b > 0$. Values of a determine the direction the function opens (positive values result in a function that is positively increasing toward an asymptotes while negative values result in a function that is decreasing toward an asymptote). Values of n determine the magnitude the function achieves from d near 0.

Note that every constant function is an affine function, every affine function a power function, and every power function a polynomial function. Of course, the reverse logic does not apply. This is important to recognize for future discussions of derivatives and their computation.